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A new operational method to solve Abel's and generalized Abel's integral equations

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ABSTRACT

Based on Jacobi polynomials, an operational method is proposed to solve the generalized Abel's integral equations (a class of singular integral equations). These equations appear in various fields of science such as physics, astrophysics, solid mechanics, scattering theory, spectroscopy, stereology, elasticity theory, and plasma physics. To solve the Abel's singular integral equations, a fast algorithm is used for simplifying the problem under study. The Laplace transform and Jacobi collocation methods are merged, and thus, a novel approach is presented. Some theorems are given and established to theoretically support the computational simplifications which reduce costs. Also, a new procedure for estimating the absolute error of the proposed method is introduced. In order to show the efficiency and accuracy of the proposed method some numerical results are provided. It is found that the proposed method has lesser computational size compared to other common methods, such as Adomian decomposition, Homotopy perturbation, Block-Pulse function, mid-point, trapezoidal quadrature, and product-integration. It is further found that the absolute errors are almost constant in the studied interval.

1. Introduction

Abel's equation is an integral equation derived directly from a concrete problem in physics, without satisfying a differential equation. This integral equation is found in the mathematical modeling of several models in physics, astrophysics, solid mechanics, scattering theory, spectroscopy, stereology, elasticity theory, and plasma physics [1–7]. It is also one of the most famous equations applied to engineering problems, including semi-conductors, heat conduction, metallurgy and chemical reactions [8,9]. In experimental physics, Abel's integral equation of the first kind finds applications in plasma diagnostics, physical electronics, nuclear physics, and optics [10,11]. The variety of applications necessitates simple and low cost by developing new methodologies to solve these equations.

Different types of Abel's integral equation have been solved using different techniques. For example, Huang et al. approximated the exact solution of the first kind Abel's integral equation by using the Taylor expansion [12]. Jahanshahi et al. used fractional integrals and Caputo fractional derivatives for solving first kind Abel's integral equations [13]. In

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[14,15], Legendre and Chebyshev wavelet methods were applied to solve Abel's integral equations of the second kind, while authors in [16] applied shifted Chebyshev polynomials to solve Abel's integral equations of this kind. Bougoffa et al. used the Adomian decomposition method for solving linear and nonlinear Abel's integral equations [17]. The Taylor expansion and Laplace transform were applied to solve Abel's integral equations of the first and second kinds [18]. Kumar and et al. introduced a new analytical method, which called Homotopy perturbation Sumudu transform method (HPSTM), to solve some fractional differential equation where appeared an Riemann–Liouville fractional integral operator [19–21]. The variational iterative method, [22], was proposed for obtaining the approximate analytic solution of Abel's Integral equations. Also, the authors in [23,24] proposed the new homotopy perturbation transform and homotopy analysis transform methods combined with the Laplace method and applied to solve some fractional differential equations. Khan and Gondal give the two-step Laplace decomposition algorithm for the solution of Abel's type singular integral equations [25,26].

In 1823, a Norwegian mathematician, N.H. Abel studied a physical problem regarding the relationship between kinetic and potential energies for falling bodies. Standard Abel's integral equations of the first and second kinds are, respectively, as follow:

$$\int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}}dt = g(x), u(x) + \int_0^x \frac{u(t)}{(x-t)^{\frac{1}{2}}}dt = f(x),$$

where $k(x,t) = 1/(x-t)^{\frac{1}{2}}$ is the singular kernel of the integral equation which makes solving this class of equations difficult. Two other problems modeled by Abel's integral equations, are presented as follow:

The oscillating pendulum: For the oscillating pendulum consider a particle with mass $m > 0$ oscillating in a potential $V(x)$ with $V(-x) = V(x)$, $V(0) = 0$, $V(x)$: strictly increasing for $x \geq 0$. For convenience, let V be differentiable at each point. Denoting by $x = x(V)$, the non-negative solution of the equation $V = V(x)$ (for $0 \leqslant V < Sup\{V(x)|\ x \in \mathbb{R}\} = \hat{V}$), the total energy of a particle is expressed as:

$$E = \frac{m}{2}x'^2 + V(x), 0 < E < \hat{V}. \tag{1}$$

The particle oscillates continuously between $x = -x(E)$ and $x = x(E)$. The period of one oscillation is defined as $T(E) = 4T^*$ where T^* is the time that the particle needs to travel from $x = 0$ to $x = x(E)$. From Eq. (1):

$$T^* = \sqrt{\frac{m}{2}}\int_0^{x(E)} \frac{dx}{\sqrt{E - V(x)}}.$$

Hence,

$$T(E) = \sqrt{8m}\int_0^{x(E)} \frac{dx}{\sqrt{E - V(x)}}, 0 \leqslant E < \hat{V}, if \quad T(E) < \infty.$$

Inserting the inverse function $x = x(V)$, the Abel integral equation is yielded as follows:

$$\int_0^E (E-V)^{-\frac{1}{2}}x'(V)dV = \frac{T(E)}{\sqrt{8m}}, 0 \leqslant E < \hat{V}.$$

By integration, $x(V)$ is obtained:

$$x(V) = \frac{1}{\pi\sqrt{8m}}\int_0^V (V-E)^{-\frac{1}{2}}T(E)dE, 0 \leqslant V < \hat{V}.$$

The cylindrical discharge: To determine the radial distribution of radiation intensity of a cylindrical discharge in plasma physics one should solve the following first kind integral equation:

$$J(\xi) = 2\int_0^{\sqrt{R^2-\xi^2}} I(\sqrt{\xi^2+\eta^2})d\eta = 2\int_\xi^R \frac{I(r)r\ dr}{\sqrt{r^2-\xi^2}}, 0 \leqslant \xi \leqslant R, \tag{2}$$

where $J(\xi)$ is the measured function and $I(r)$ is the one to be determined (Fig. 1). By using the following variables and functions:

$$x = R^2 - \xi^2, y = R^2 - r^2, F(x) = J(\sqrt{R^2 - x}), f(y) = I(\sqrt{R^2 - y}),$$

Eq. (2) reduces to the standard Abel integral equation for functions F and f:

$$F(x) = \int_0^x \frac{f(y)dy}{\sqrt{x-y}}.$$

The difficulty in solving these equations demands a simple approach to approximate and solve the various types of Abel's integral equations. To achieve this objective, in this study, an operational Jacobi collocation method and Laplace transform are merged to establish a new methodology. Using Laplace transform simplifies the singular kernel of Abel's integral equations. By applying the resultant matrix relations, Abel's equations are equated to a system of either linear or nonlinear algebraic equations.