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On the data-driven COS method

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ABSTRACT

In this paper, we present the data-driven COS method, ddCOS. It is a Fourier-based financial option valuation method which assumes the availability of asset data samples: a characteristic function of the underlying asset probability density function is not required. As such, the presented technique represents a generalization of the well-known COS method [1]. The convergence of the proposed method is $\mathcal{O}(1/\sqrt{n})$, in line with Monte Carlo methods for pricing financial derivatives. The ddCOS method is then particularly interesting for density recovery and also for the efficient computation of the option's sensitivities Delta and Gamma. These are often used in risk management, and can be obtained at a higher accuracy with ddCOS than with plain Monte Carlo methods.

1. Introduction

In quantitative finance, statistical distributions are commonly used for the valuation of financial derivatives and within risk management. The underlying assets are often modelled by means of stochastic differential equations (SDEs). Except for the classical and most simple asset models, the corresponding *probability density function* (PDF) and *cumulative distribution function* (CDF) are typically not known and need to be approximated.

In order to compute option prices, and to approximate statistical distributions, Fourier-based methods are commonly used numerical techniques. They are based on the connection between the PDF and the *characteristic function* (ChF), which is the Fourier transform of the probability density. The ChF is often available, and sometimes even in closed form, for the broad class of regular diffusions and also for Lévy processes. Some representative efficient Fourier pricing methods include those by Carr and Madan [2], Boyarchenko and Levendorskii [3], Lewis [4] and Fang and Oosterlee [1]. Here, we focus on the COS method from [1], which is based on an approximation of the PDF by means of a cosine series expansion.

Still, however, the asset dynamics for which the ChF are known is not exhaustive, and for many relevant asset price processes we do not have such information to recover the density. In recent years several successful attempts have been made to employ Fourier pricing methods without the explicit knowledge of the ChF. In Grzelak and Oosterlee [5], for example, a hybrid model with stochastic volatility and stochastic interest rate was linearized by means of expectation operators to cast the approximate system of SDEs in the framework of affine diffusions. Ruijter and Oosterlee [6] discretized the governing asset SDEs first and then worked with the ChF of the discrete asset process, within the framework of the COS method. Borovykh et al. [7] used the Taylor expansion to derive a ChF for which they could even price Bermudan options highly

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efficiently. In this work, we extend the applicability of the COS method to the situation where only data (samples from an unknown distribution) are available.

The density estimation problem, using a *data-driven* PDF, has been intensively studied in the last decades, particularly since it is a component in the *machine learning* framework [8]. Basically, density estimators can be classified into parametric and non-parametric estimators. The first type relies on the fact that prior knowledge is available (like moments) to determine the relevant parameters, while for non-parametric estimators the parameters need to be determined solely from the samples themselves. Within this second type of estimators we can find histograms, kernel density and orthogonal series estimators. A thorough description of these estimators is provided in [9]. More recently, some applications in finance have also appeared, see [10–12], for example.

For the valuation of financial derivatives, we will combine density estimators with Fourier-based methods, so orthogonal series form a natural basis. We will focus on the framework of *statistical learning*, see [13]. In statistical learning, a regularization is employed to derive an expression for the data-driven empirical PDF. By representing the unknown PDF as a cosine series expansion, a closed-form solution of the regularization problem is known [13], which forms the basis of the *data-driven COS method* (ddCOS). However, in order to employ the COS method machinery, underlying risk-neutral asset samples are required, i.e. they need to be generated according to some underlying model. This fact implies that the technique presented here results in a hybrid Monte Carlo-Fourier method.

The use of the COS method gives us expressions for option prices and, in particular, for the *option sensitivities or Greeks*. These option Greeks are the derivatives of option price with respect to a variable or parameter. The efficient computation of the Greeks is a challenging problem when only asset samples are available. Existing approaches are based on Monte Carlo-based techniques, like on finite-differences (bump and revalue), pathwise or likelihood ratio techniques, for which details can be found in [14], chapter 7. Several extensions and improvements of these approaches have appeared, for example, based on adjoint formulations [15], the ChF [16,17], Malliavin calculus [18,19], algorithmic differentiation [20,21] or combinations of these [22–24]. Intuitively, the ddCOS method follows a similar approach as the likelihood ratio method, i.e. it relies on the differentiation of the (recovered) density function. On the other hand, our method can be also related to the improved methodologies employing the so-called *Malliavin derivative*, since it introduces a sample-based *weighted coefficients* that multiply the payoff coefficients. For both techniques, the differentiation of the payoff function (or payoff coefficients) is avoided.

All in all, the computation of the Greeks can be quite involved. The ddCOS method is not directly superior to Monte Carlo methods for option valuation, but it is competitive for the computation of the corresponding sensitivities. We derive simple expressions for the Greeks Delta and Gamma. The importance of Delta and Gamma in dynamic hedging and risk management is well-known. A useful application is found in the *Delta–Gamma approach* [25] to quantify market risk. The approximation of risk measures like *Value-at-Risk* (VaR) and *Expected Shortfall* (ES) under the Delta–Gamma approach is still nontrivial. Next to Monte Carlo methods, Fourier techniques have been employed in this context, when the ChF of the change in the value of the option portfolio is known (see [26,27]). For example, the COS method has been applied in [28] to efficiently compute the VaR and ES under the Delta–Gamma approach. The ddCOS method may generalize the applicability to the case where only data is available.

This paper is organized as follows. The ddCOS method, and the origins in statistical learning and Fourier-based option pricing, are presented in Section 2. Variance reduction techniques can also be used within the ddCOS method, providing an additional convergence improvement. We provide insight and determine values for the method's open parameters in Section 3. Numerical experiments, with a focus on the option Greeks, are presented in Section 4. We conclude in Section 5.

2. The data-driven COS method

In this section we will discuss the ddCOS method, in which aspects of the Monte Carlo method, density estimators and the COS method are combined to approximate, in particular, the option Greeks Delta and Gamma. We will focus on European options here.

The COS method in [1] is a Fourier-based method by which option prices and sensitivities can be computed for various options under different models. The method relies heavily on the availability of the ChF, i.e., the Fourier transform of the PDF. In the present work, we assume that only asset samples are available, not the ChF, resulting in the data-driven COS method. It is based on regularization in the context of the statistical learning theory, presented briefly in Section 2.2. The connection with the COS method is found in the fact that the data-driven PDF appears as a cosine series expansion.

2.1. The COS method

The starting point for the well-known COS method is the risk-neutral option valuation formula, where the value of a European option at time t, $v(x,t)$, is an expectation under the risk neutral pricing measure, i.e.,

$$v(x,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}[v(y,T)|x] = \mathrm{e}^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\mathrm{d}y, \tag{1}$$