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Approximation by Kantorovich form of modified Szász–Mirakyan operators

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ABSTRACT

In the present article, we consider the Kantorovich type generalized Szász–Mirakyan operators based on Jain and Pethe operators [32]. We study local approximation results in terms of classical modulus of continuity as well as Ditzian–Totik moduli of smoothness. Further we establish the rate of convergence in class of absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation.

1. Introduction

In 1977, Jain and Pethe [32] generalized the well-known Szász–Mirakyan operators [40] as:

$$\begin{aligned} S_n^{[\alpha]}(f;x) &= (1+n\alpha)^{-\frac{x}{\alpha}}\sum_{k=0}^{\infty}\left(\alpha+\frac{1}{n}\right)^{-k}\frac{x^{(k,-\alpha)}}{k!}f\left(\frac{k}{n}\right) \\ &= \sum_{k=0}^{\infty}s_{n,k}^{[\alpha]}(x)f\left(\frac{k}{n}\right), \end{aligned} \tag{1.1}$$

where

$$s_{n,k}^{[\alpha]}(x) = (1+n\alpha)^{-\frac{x}{\alpha}}\left(\alpha+\frac{1}{n}\right)^{-k}\frac{x^{(k,-\alpha)}}{k!},$$

$x^{(k,-\alpha)} = x(x+\alpha)\dots(x+(k-1)\alpha)$, $x^{(0,-\alpha)} = 1$ and f is any function of exponential type such that

$$|f(t)| \le Ke^{At} \quad (t \ge 0),$$

for some finite constants $K, A > 0$. Here $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is such that

$$0 \le \alpha_n \le \frac{1}{n}.$$

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The operators $S_n^{[\alpha]}$ have also been considered by Stancu [39], Mastroianni [36], Della Vecchia and Kocic [17] and Finta [26,27].

Abel and Ivan [1] gave the following alternate form of operators (1.1) (by putting $c=\frac{1}{n\alpha}$):

$$S_{n,c}(f;x)=\sum_{k=0}^{\infty}\left(\frac{c}{1+c}\right)^{ncx}\binom{ncx+k-1}{k}(1+c)^{-k}f\left(\frac{k}{n}\right),\qquad x\geq 0,\tag{1.2}$$

where $c=c_n\geq\beta(n=0,1,2,......)$ for certain constant $\beta>0$. Also, for a particular case $\alpha=\frac{1}{n}$, the operators (1.1) reduce to another form, which was considered by Agratini [2] as follows:

$$S_n^{[\frac{1}{n}]}(f;x)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right),\tag{1.3}$$

where

$$(nx)_k=nx(nx+1)\dots(nx+k-1),\qquad k\geq 1,$$

and $(nx)_0=1$. These operators (1.3) are special cases of Lupaş operators [35]. The operators (1.3) have also been studied in [25] and [37].

Agratini [3] modified the operators (1.3) into integral form in Kantorovich sense as:

$$T_n(f;x)=n\sum_{k=0}^{\infty}2^{-nx}\frac{(nx)_k}{2^k k!}\int_{k/n}^{(k+1)/n}f(t)dt,\tag{1.4}$$

and studied some approximation properties. Very recently, Deo et al. considered generalized positive linear operators based on Pólya–Eggenberger and inverse Pólya–Eggenberger distribution in [21] and furthermore, they gave Kantorovich variant of these generalized operators in [18]. Several researchers have given some interesting results on Kantorovich variant of various operators (see [7–16,30,38]). Motivated by above works, for any bounded and integrable function f defined on $[0,\infty)$, we also modify the operators (1.1) in Kantorovich form:

$$L_n^{[\alpha]}(f;x)=n\sum_{k=0}^{\infty}s_{n,k}^{[\alpha]}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt.\tag{1.5}$$

Special cases:

(1) For $\alpha=0$ in (1.5), we get Szász–Kantorovich operators given by Totik in [41].
(2) For $\alpha=\frac{1}{n}$ in (1.5), we obtain another Kantorovich operators considered by Agratini [3].

The focus of this paper is to study the approximation properties of modified Kantorovich operators (1.5). First we obtain local approximation formula via modulus of continuity of second order then we use Ditzian–Totik moduli of smoothness to discuss the rate of convergence of our operators. Finally, we establish the rate of convergence for functions having derivatives of bounded variation. The properties discussed in this article can be found in some recent papers like [4,6,19,20,24,28,29,31,33].

2. Auxiliary results

In order to prove the main convergence properties of operators (1.5), we need the following basic results:

Lemma 2.1 *[39]. For the generalized Szász–Mirakyan operators (1.1) hold*

$$S_n^{[\alpha]}(1;x)=1,\qquad S_n^{[\alpha]}(t;x)=x,$$

and

$$S_n^{[\alpha]}\left(t^2;x\right)=x^2+\left(\alpha+\frac{1}{n}\right)x.$$

Proposition 2.1. *For the operators (1.1), there hold the following higher order moments:*

$$S_n^{[\alpha]}\left(t^3;x\right)=x^3+3\left(\alpha+\frac{1}{n}\right)x^2+\left(2\alpha^2+\frac{3\alpha}{n}+\frac{1}{n^2}\right)x,$$

and

$$S_n^{[\alpha]}\left(t^4;x\right)=x^4+6\left(\alpha+\frac{1}{n}\right)x^3+\left(11\alpha^2+\frac{18\alpha}{n}+\frac{7}{n^2}\right)x^2+\left(6\alpha^3+\frac{12\alpha^2}{n}+\frac{7\alpha}{n^2}+\frac{1}{n^3}\right)x.$$

Proof. By definition we can write

$$S_n^{[\alpha]}\left(t^3;x\right)=(1+\alpha n)^{-\frac{x}{\alpha}}\sum_{k=1}^{\infty}\frac{x(x+\alpha)\dots(x+(k-1)\alpha)n^k}{k!(1+\alpha n)^k}\frac{k^3}{n^3}$$