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# Computation of control for linear approximately controllable system using weighted Tikhonov regularization

Ravinder Katta [a,*], G.D. Reddy [b], N. Sukavanam [a]

[a] Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, Uttarakhand 247667, India
[b] Department of Mechanical and Aerospace Engineering, Indian Institute of Technology Hyderabad, Kandi, Sangareddy, Telangana 502285, India

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ABSTRACT

Computing steering control for an approximately controllable linear system for a given target state is an ill-posed problem. We use a weighted Tikhonov regularization method and compute the regularized control. It is proved that the target state corresponding to the regularized control is close to the actual state to be attained. We also obtained error estimates and convergence rates involved in the regularization procedure using both the a priori and a posteriori parameter choice rule. Theory is substantiated with numerical experiments.



## 1. Introduction

Controllability is one of the important qualitative properties of a dynamical system which plays a crucial role in the development of the modern mathematical control theory. The concept of controllability was introduced by Kalman et al. [8]. Systematic study of controllability was started in the 1960s and since then controllability of various kinds of systems has been studied extensively by many researchers (see [1,2,10,16]). Controllability generally means that it is possible to steer the dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Approximate controllability means that there exists a control which steers the control system from an arbitrary initial state to an arbitrary neighborhood of the given target state. Approximately controllable systems are more prevalent and they have many potential applications, especially in some problems arising in aerospace engineering, diffusion systems, nuclear reactors, robotics, and missiles etc.

Conditions for approximate and exact controllability for linear infinite dimensional systems were discussed in the monograph [11]. Klamka [9,12,13] formulated sufficient conditions for exact and approximate constrained controllability assuming that the values of controls are in a convex and closed cone with the vertex at zero. Klamka [14] and Respondek [20] established necessary and sufficient conditions for constrained approximate controllability for linear dynamical systems described by abstract differential equations with an unbounded control operator. In this paper, we present a new approach for computing control for an unconstrained approximately controllable linear system in the context of ill posed problems using weighted Tikhonov regularization. The control obtained in this paper is not restricted to take on values in a preassigned subset of the control space.

* Corresponding author.
*E-mail addresses:* kravidma@iitr.ac.in, kattaravinder11@gmail.com (R. Katta), damu@iith.ac.in (G.D. Reddy), nsukvfma@iitr.ac.in (N. Sukavanam).

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Let $V$ and $U$ be Hilbert spaces called state and control spaces, respectively. Let $X = L_2(J, V)$ and $Y = L_2(J, U)$ be function spaces, where $J := [t_0, \tau] \subseteq [0, \infty)$. We shall use the notation $\langle \cdot, \cdot \rangle$ for the inner product on a Hilbert space and $\| \cdot \|$ for the corresponding norm.

Consider the linear control system

$$\begin{aligned} \frac{dx}{dt} &= Ax(t) + Bu(t), \\ x(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where $A$: $D(A) \subseteq V \to V$ is a densely defined closed linear operator on $V$ that generates a $C_0$ semigroup $T(t),\ t \geq 0$ and $B$: $U \to V$ is a bounded linear operator.

Recall (see, [17]) that, for $u \in Y$, the *mild solution* of (1.1) is given by

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - r)Bu(r)dr. \tag{1.2}$$

The control system (1.1) is said to be *exactly controllable* if for every $x_0$ and $x_\tau \in V$, there exists $u \in Y$ such that the mild solution $x \in X$ satisfies the condition $x(\tau) = x_\tau$.

The control system (1.1) is said to be *approximately controllable* if for every $\epsilon > 0$ and for every $x_0$ and $x_\tau \in V$, there exists $u \in Y$ such that

$$x(\tau) := T(\tau - t_0)x_0 + \int_{t_0}^{\tau} T(\tau - r)Bu(r)dr \tag{1.3}$$

satisfies

$$\|x(\tau) - x_\tau\| \leq \epsilon. \tag{1.4}$$

In other words the control system (1.1) is said to be *approximately controllable* if the reachable set of (1.1) namely,

$$R_0 := \{x(\tau) \in V : x \in X \text{ is a mild solution of (1.1) corresponding to the control } u \in Y\}$$

is dense in $V$.

Let $L$: $Y \to V$ be the bounded linear operator defined by

$$Lu = \int_{t_0}^{\tau} T(\tau - r)Bu(r)dr, \ \ u \in Y. \tag{1.5}$$

Then the problem of finding a control $u$ for a given target state $x_\tau \in V$ satisfying the requirements (1.3) and (1.4) is equivalent to the problem of solving the operator equation

$$Lu = v, \tag{1.6}$$

where

$$v = x_\tau - T(\tau - t_0)x_0.$$

Assume that the system (1.1) is approximately controllable. Then the problem of solving the operator equation (1.6) is ill-posed in the sense of Hadamard [6], that is, small perturbations in $v$ can produce large deviations in the solution, which is not desirable for practical purposes. Therefore, it is not advisable to solve (1.6) directly to obtain $u$. Hence a regularization method has to be applied for finding stable approximate solution of (1.6). The authors of [24] have been constructed such a family for a linear approximately controllable system using Tikhonov regularization and the theory of ill-posed problems, and also developed a method to select the regularization parameter which yielded some error estimates involved in the regularization procedure.

Tikhonov regularization is one of the famous classical regularization methods for solving ill-posed problems (see [4,19,23,25]).

It is known that the Tikhonov regularization method makes the regularized solution very smooth which results in the loss of many details of the desired solution. Hence we apply the weighted Tikhonov regularization method [5,7,15] for finding stable approximate solutions of (1.6). In weighted Tikhonov regularization one solves the well-posed operator equation

$$\left((L^*L)^{\frac{\alpha+1}{2}} + \beta I\right)u = (L^*L)^{\frac{\alpha-1}{2}} L^*v \tag{1.7}$$

and solution of (1.7), to be denoted by $u_{\beta,\,\alpha}$ is the unique minimizer of the functional

$$u \mapsto \|Lu - v\|_w^2 + \beta\|u\|_Y \tag{1.8}$$

with $\|v\|_w = \|(LL^*)^{\frac{\alpha-1}{4}} v\|_V$. Here $L^*$ is the unique adjoint of $L$ and $\beta > 0$ is a regularization parameter and $0 < \alpha \leq 1$ is a positive constant. $\beta$ plays a crucial role in obtaining error estimates and convergence rates involved in the regularization procedure whereas $\alpha$ controls the smoothness of the regularized solution.