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Connectivity of half vertex transitive digraphs☆

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ABSTRACT

A bipartite digraph is said to be a half vertex transitive digraph if its automorphism acts transitively on the sets of its bipartition, respectively. In this paper, bipartite double coset digraphs of groups are defined and it is shown that any half vertex transitive digraph is isomorphic to some half double coset digraph, and we show that the connectivity of any strongly connected half transitive digraph is its minimum degree.

1. Introduction

A graph (digraph) is said to be *vertex transitive* if its automorphism group acts transitively on its vertex set, and is *edge (arc) transitive* if its automorphism group acts transitively on its edge (arc) set. There is close relation between the graph symmetry and its connectivity. For instance, the edge connectivity of a connected vertex transitive graph is its regular degree [4] and the connectivity of a connected edge transitive graph is its minimum degree [7]. Similar results also hold for some digraphs [1,2]. These motivate us to study the connectivity properties of graphs (digraphs) with k vertex or edge orbits under the action of their automorphism groups. The first step under this direction is to study the connectivity properties of graphs (digraphs) with two orbits.

Let $D = (V, E)$ be a digraph, and $F \subseteq V$. Set

$$N^+(F) = \{v \in V \setminus F : \text{ there exists } u \in F \text{ satisfying } (u, v) \in E\},$$
$$N^-(F) = \{v \in V \setminus F : \text{ there exists } u \in F \text{ satisfying } (v, u) \in E\}.$$

Set $d_D^+(u) = |N^+(u)|$, and $d_D^-(u) = |N^-(u)|$, which are called *out-degree* and *in-degree* of u, respectively. Set $\delta^+(D) = \min\{d_D^+(u) : u \in V(D)\}$, $\delta^-(D) = \min\{d_D^-(u) : u \in V(D)\}$, and $\delta(D) = \min\{\delta^-(D), \delta^+(D)\}$, which are called the *minimum out-degree*, the *minimum in-degree* and the *minimum degree* of D, respectively. The *arc connectivity* $\lambda(D)$ is the minimum cardinality of all arc sets S in D such that $D - S$ is not strongly connected, and the *vertex connectivity* $\kappa(D)$, simply *connectivity*, is the minimum cardinality of all vertex sets T in D such that $D - T$ is not strongly connected. The following result is well known:

$$\kappa(D) \leq \lambda(D) \leq \delta(D).$$

Thus, if $\kappa(D) = \delta(D)$, then $\lambda(D) = \delta(D)$. D is said to be strongly connected if $\kappa(D) \geq 1$.

For results on the connectivity of vertex transitive or arc transitive digraphs, see the excellent survey [7] and papers [3,5,6]. Notation and definitions not defined here are referred to [8,9].

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2. Half vertex transitive digraphs and half double coset digraphs

Let G be a group and $S \subseteq G \backslash \{1\}$, where 1 is the identity element of G. The *Cayley digraph* $D = D(G, S) = (V(D), E(D))$ is a directed graph with vertex set $V(D) = G$ and arc set $E(D) = \{(x, y) : \text{there exists } s \in S \text{ satisfying } yx^{-1} = s \text{ for } x, y \in G\}$. If S is inverse-closed, that is, $S = S^{-1}$, then $(x, y) \in E(D)$ if and only if $(y, x) \in E(D)$. Thus, D corresponds to an undirected graph $C(G, S)$, called a *Cayley graph*. For $a \in G$, the *right multiplication* $R(a)$: $g \to ga$, $g \in G$, is clearly an automorphism of any Cayley digraph of G. *Automorphism group* $Aut(D)$ of D is the set of all automorphism of the digraph D. Let $R(G) = \{R(a) : a \in G\}$. Then $R(G)$ is a subgroup of $Aut(D)$ and it acts transitively on the vertices of D. Thus Cayley digraphs (graphs) are vertex transitive.

Definition 1. Let $D = (U, W; E)$ be a bipartite digraph with bipartition $U \cup W$. If $Aut(D)$ acts transitively on U and W, respectively, then we call D a half vertex transitive digraph.

The undirected version of half vertex transitive digraph is due to [9]. Let G be a group, R be a subgroup of G, we use $[G$: $R]$ to denote the set of all right cosets of R in G, that is $[G : R] = \{Rg : g \in G\}$.

Definition 2. Let G be a group, L and R be two subgroups of G. Assume T_i is a union of some double cosets of the form RgL for $i = 1, 2$. Define the half double coset digraph $D = D(G, L, R; T_1, T_2) = (V(D), E(D))$ as follows:

(i) $V(D) = [G : L] \cup [G : R]$,
(ii) $(Lg, Rg') \in E(D)$ if and only if $Rg' = Rt_1 g$ for some $t_1 \in T_1$,
(iii) $(Rg, Lg') \in E(D)$ if and only if $Rg = Rt_2 g'$ for some $t_2 \in T_2$.

For half double coset digraphs, we have the following theorem.

Theorem 2.1. *Let $D = D(G, L, R; T_1, T_2)$ be a half double coset digraph. Then*

(i) *$R(G) \leq Aut(D)$, thus D is half vertex transitive, where $R(G) = \{R(a) : a \in G\}$,*
(ii) *$d^+_D(Lg) = ||[T_1 : R]||$, $d^-_D(Lg) = ||[T_2 : R]||$, $d^+_D(Rg) = ||[T_2^{-1} : L]||$ and $d^-_D(Rg) = ||[T_1^{-1} : L]||$, where $||[T_i: R]||$ and $||[T_i^{-1} : L]||$ denote the number of right cosets of R in T_i and L in T_i^{-1} for $i = 1, 2$, respectively,*
(iii) *D is strongly connected if and only if $G = \langle T_2^{-1} T_1 \rangle$.*

Proof. For any $R(a) \in R(G)$ and $(Lg, Rg') \in E(D)$, there exists $t_1 \in T_1$ satisfying $Rg' = Rt_1 g$. Thus $(Lg)^{R(a)} = Lga$, $(Rg')^{R(a)} = Rt_1 ga$. By Definition 2, $((Lg)^{R(a)}, (Lg')^{R(a)}) \in E(D)$. (i) follows. To prove (ii), it suffices to note that $N^+(L) = \{Rt_1 : t_1 \in T_1\}$, $N^-(L) = \{Rt_2 : t_2 \in T_2\}$, $N^+(R) = \{Lt_2^{-1} : t_2 \in T_2\}$ and $N^-(R) = \{Lt_1^{-1} : t_1 \in T_1\}$. Clearly, D is strongly connected if and only if there exists a directed path from L to Lg for any $g \in G$, that is, there exists an integer k and $t_i^{(1)} \in T_1$ and $t_i^{(2)} \in T_2$ for $i \in \{1, 2, \ldots, k\}$ satisfying

$$L \to Rt_1^{(1)} \to L\left(t_1^{(2)}\right)^{-1} t_1^{(1)} \to \cdots \to L\left(t_k^{(2)}\right)^{-1} t_k^{(1)} \cdots \left(t_1^{(2)}\right)^{-1} t_1^{(1)} = Lg,$$

the above equation holds if and only if $g \in \langle T_2^{-1} T_1 \rangle$. (iii) follows. □

Theorem 2.2. *Let $D = (U, W; E)$ be a half vertex transitive digraph. Let $G = Aut(D)$, $L = G_u$ and $R = G_w$, where $u \in U$, $w \in W$, and G_u and G_w are the stabilizers of u and w in G, respectively. Set $T_1 = \{g \in G : w^g \in N^+(u)\}$ and $T_2 = \{g \in G : w^g \in N^-(u)\}$. Then $D \cong D(G, L, R; T_1, T_2)$.*

Proof. We prove a sequences of claims from which the results follow.

Claim 1. *T_1 and T_2 are unions of some double cosets of the form RgL.*

In fact, for any $g \in T_1$, $w^g \in N^+(u)$, that is, $(u, w^g) \in E(D)$. Then for any $l \in L$, $(u, w^g)^l = (u, w^{gl}) \in E(D)$. Since for any $r \in R$, $w^r = w$, thus $(u, w^{rgl}) \in E(D)$. By definition, $rgl \in T_1$, and therefore $RgL \subseteq T_1$. Similarly, if $g \in T_2$, then $RgL \subseteq T_2$.

Let $D_1 = D(G, L, R; T_1, T_2)$, $U = \{u_1 = u, u_2, \ldots, u_m\}$, and $W = \{w_1 = w, w_2, \ldots, w_n\}$. Since D is half vertex transitive, there exists $g_i^{(1)}, g_j^{(2)} \in G$ satisfying $u^{g_i^{(1)}} = u_i$ and $w^{g_j^{(2)}} = w_j$ for $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. Define a map σ from $V(D)$ to $V(D_1)$ as follows,

$$\begin{aligned} \sigma \ : \ u_i &\to Lg_i^{(1)} \qquad i = 1, 2, \ldots, m, \\ w_j &\to Rg_j^{(2)} \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Claim 2. *σ is bijective.*

For any $Lg \in [G$: $L]$, suppose that $u^g = u_i$ for $i \in \{1, 2, \ldots, m\}$, then $u^g = u^{g_i^{(1)}}$, that is $u^{g_i^{(1)} g^{-1}} = u$. Thus $g_i^{(1)} g^{-1} \in L$ and $Lg = Lg_i^{(1)} = u_i^\sigma$. Similarly, for any $Rg \in [G$: $R]$, if $w^g = w_s$ for $s \in \{1, 2, \ldots, n\}$, then $Rg = Rg_s^{(2)} = w_s^\sigma$. If $u_i^\sigma = u_j^\sigma$ for $1 \leq i < j \leq m$, then $Lg_i^{(1)} = Lg_j^{(1)}$, and so there exists $l \in L$ satisfying $g_j^{(1)} = lg_i^{(1)}$. Thus $u_j = u^{g_j^{(1)}} = u^{lg_i^{(1)}} = u^{g_i^{(1)}} = u_i$. Similarly, if $w_s^\sigma = w_t^\sigma$ for $1 \leq s < t \leq n$, then $w_s = w_t$.