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# $C^2$ positivity-preserving rational interpolation splines in one and two dimensions

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ABSTRACT

A class of rational quartic/cubic interpolation spline with two local control parameters is presented, which can be $C^2$ continuous without solving a linear system of consistency equations for the derivative values at the knots. The effects of the local control parameters on generating interpolation curves are illustrated. For $C^2$ interpolation, the given interpolant can locally reproduce quadratic polynomials and has $O(h^2)$ or $O(h^3)$ convergence. Simple schemes for the $C^2$ interpolant to preserve the shape of 2D positive data are developed. Moreover, based on the Boolean sum of quintic interpolating operators, a class of bi-quintic partially blended rational quartic/cubic interpolation surfaces is also constructed. The given interpolation surface provides four local control parameters and can be $C^2$ continuous without using the second or higher mixed partial derivatives on a rectangular grid. Simple sufficient data dependent constraints are also derived on the local control parameters to preserve the shape of a 3D positive data set arranged over a rectangular grid.



## 1. Introduction

The problem of modeling interpolation curves and surfaces to given data has been studied with various requirements, such as the smoothness of the interpolation curves and surfaces, the preservation of the shape features of the data, the computational complexity, and so on. In the present paper, we are particularly concerned with the construction of positivity-preserving interpolation curves and surfaces with $C^2$ continuity. Positivity is one of the essential shape features. Many physical situations have entities that gain meaning only when their values are positive, such as a probability distribution function, monthly rainfall amounts, speed of winds at different intervals of time, and half-life of a radioactive substance. For generating visually pleasing interpolation curves and surface, smoothness is a significant factor. Generally speaking, for most applications, $C^1$ continuity is sufficient. Sometimes, however, curvature continuity is needed and this leads to the need for $C^2$ continuity.

Piecewise rational cubic spline methods have been widely studied and developed for constructing $C^2$ positivity-preserving interpolation curves, see for example [1,2] and the references quoted therein. In [1], a class of $C^2$ rational cubic/linear spline with two local control parameters was proposed for constructing positivity-preserving interpolation curves though 2D positive data. By using a rational cubic/quadratic spline with three local control parameters, a scheme for generating $C^2$ positivity-preserving interpolation curves for 2D positive data was proposed in [2]. These methods have a common feature that in order to reach the desired $C^2$ continuity, the solution of a global linear system of consistency equations for the

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derivative values at the knots is required. Therefore, changes to any interpolation data or control parameters will require solving again all the equations.

There are some $C^2$ interpolation curve modelings based on rational quartic and quintic splines. In [3], a rational quartic/linear interpolation spline with two local control parameters was proposed. It was shown that by choosing special derivative values at the knots, the rational quartic/linear interpolation spline can be $C^2$ continuous. However, for the rational quartic/linear interpolation spline proposed in [3], the proper choice of the control parameters to guarantee a $C^2$ positivity-preserving interpolation spline is still an open question. In [4], Han gave an explicit representation of a $C^2$ rational quartic/cubic interpolation spline with a local control parameters, which can preserve the local convexity properties of the given 2D data by appropriate choosing choice of control parameters. Nevertheless, the rational quartic/cubic interpolation spline has some disadvantages such as it does not have positivity-preserving property as well as monotonicity-preserving property in general. In [5,6], these shortcomings were overcome by introducing additional local control parameters into the rational quartic/cubic interpolation spline given in [4]. Recently, a $C^2$ piecewise rational quartic/quadratic interpolant with a control parameter was described in [7], which has positivity-preserving property for 2D positive data. But the changes of a local control parameter will change three associated curve segments. In [8], a $C^2$ rational quintic/quartic Hermite spline with a local control parameter was designed and can be used to construct positive curves through 2D positive data by properly constraining the local control parameter.

For visualizing 3D positive data given on a rectangular grid, some $C^1$ positivity-preserving bivariate interpolation splines with local control parameters have been proposed by using the Coons surface technique developed in [9], see for example [10–13] and the references quoted therein. In [10] and [11], by replacing the classical cubic Hermite interpolation basis for the classical bi-cubic Coons surface with different kinds of rational cubic Hermite-type interpolation basis, two classes of $C^1$ rational bi-cubic functions were presented. And constraints concerning the local control parameters were given for visualizing 3D positive data on rectangular grid. Like the classical bi-cubic Coons surface technique, these schemes need to provide the cross-boundary derivatives or the twists on rectangular grid for generating interpolation surfaces. In [12,13], based upon the Boolean sum of cubic interpolating operators, by blending different rational cubic interpolation splines as the boundary functions, simpler schemes without making use of twists for constructing $C^1$ positive-preserving interpolation surfaces were given. This transfinite interpolation method is convenient since it is possible to control the shape of the interpolation surfaces by using the boundary functions, though it has to pay the price that the generated interpolation surfaces have zero twist vectors at the data points.

By using the classical Coons surface technique, however, it is a more difficult task to construct $C^2$ positivity-preserving interpolation surfaces for 3D positive data defined over a rectangular grid and there are few publications concerning this topic, see [14] and [15]. In [14], by taking the Boolean sum of two rational cubic/quadratic Hermite-type blending functions and solving two linear systems of equations with respect to the first partial derivative values on a rectangular grid, it was shown that there exist rational bi-cubic spline interpolants of the continuity class $C^2$ which are S-convex, monotone, or positive if the data sets have these properties. In [15], it was proved that by constraining the first and higher partial derivatives together with the second and higher mixed partial derivatives on rectangular grid, positivity can be always preserved by $C^2$ bi-quartic interpolation splines and monotonicity by $C^2$ bi-quintic interpolation splines. In practical applications, the second and higher mixed partial derivatives are hard to estimate and control, and there may also exist compatibility problems in generating the classical $C^2$ bi-quintic Coons surface, see [16]. Recently, a class of rational bi-quintic interpolation spline with three local control parameters was constructed in [17]. For generating interpolation surfaces, the given interpolant only uses the values of the interpolated function and can be $C^2$ continuous for equally spaced knots. However, the important shape preserving properties of the rational bi-quintic interpolant have not been discussed and thus there is still some work to be done.

The purpose of this paper is to present a rational quartic/cubic Hermite-type interpolation basis with two control parameters, which can be used to construct $C^2$ interpolation curves without solving a linear system of consistency equations for the derivative values at the knots and includes the interpolation splines developed in [18,19] as special cases. The control parameters have foreseeable shape control effects on the generated $C^2$ interpolation curves and the changes of a local control parameter will only affect two curve segments. Simple constraint conditions concerning the local control parameters for the $C^2$ interpolation curves to preserve the shape of 2D positive data are developed. Compared with the positivity-preserving conditions developed in [5,6], the given positivity-preserving conditions are simpler and easier to implement. Moreover, based on the Boolean sum of quintic interpolating operators, by blending together the new proposed rational quartic/cubic interpolation splines as the boundary functions, a class of bi-quintic partially blended rational quartic/cubic interpolation surfaces with four local control parameters is also constructed. The given interpolation surface can be $C^2$ continuous without using the second or higher mixed partial derivatives on a rectangular grid. Simple sufficient data dependent constraints are also deduced on local control parameters to generate $C^2$ positivity-preserving interpolation surfaces for 3D positive data given on a rectangular grid.

The rest of this paper is organized as follows. Section 2 gives the construction of the new $C^2$ rational quartic/cubic interpolation spline with two local control parameters. The corresponding convergence analysis is shown. The shape effects of the control parameters on generating interpolation curves are given. Simple schemes for generating positivity-preserving interpolation curves are developed. In Section 3, a $C^2$ bi-quintic partially blended rational quartic/cubic interpolation surface with four local control parameters is described. Sufficient conditions for constructing positivity-preserving interpolation surfaces are discussed in detail. Conclusions are given in Section 4.