Applied Mathematics and Computation 316 (2018) 296–309

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

How many k-step linear block methods exist and which of them is the most efficient and simplest one?

Higinio Ramos [a,b,*], Paul Popescu [c]

[a] Scientific Computing Group, Universidad de Salamanca, Plaza de la Merced, Salamanca 37008, Spain
[b] Escuela Politécnica Superior de Zamora, Campus Viriato, Zamora 49022, Spain
[c] Department of Applied Mathematics, University of Craiova, Str. A. I. Cuza, 13, 200585 Craiova, Romania

ARTICLE INFO

Keywords:
Ordinary differential equations
Initial value problems
k-step block methods
Efficient formulation

ABSTRACT

There have appeared in the literature a lot of k-step block methods for solving initial-value problems. The methods consist in a set of k simultaneous multistep formulas over k non-overlapping intervals. A feature of block methods is that there is no need of other procedures to provide starting approximations, and thus the methods are self-starting (sharing this advantage of Runge–Kutta methods). All the formulas are usually obtained from a continuous approximation derived via interpolation and collocation at $k+1$ points. Nevertheless, all the k-step block methods thus obtained may be considered as different formulations of one of them, which results to be the most efficient and simple formulation of all of them. The theoretical analysis and the numerical experiments presented support this claim.

1. Introduction

Consider a first-order initial value problem (I.V.P.) of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{1}$$

on a given interval $[x_0, b] \in \mathbb{R},$ where conditions about the existence of a unique solution are assumed.

Among the numerical methods available in literature for solving the problem in (1) are the block methods. Block methods were proposed firstly by Milne [1]. They have the advantages of being more efficient in terms of cost implementation, time of execution and accuracy, and were developed to tackle some of the setbacks of predictor-corrector methods [2–7]. The block methods contain main and additional methods, a concept that is due to Brugnano and Trigiante [8]. They have appeared in literature dozens of block methods. This paper aims at analyzing and classifying these methods to show that most of them are the same. In fact, we will see that for each $k \in \mathbb{N}, k \geq 2,$ there is only one k-step method that is the simplest one.

The paper is organized as follows. In Section 2, we made a detailed analysis of 2-step block methods, showing that different methods appeared in literature in fact correspond to different formulations of the same method. Among these formulations there is only one which is the most efficient in terms of computational cost. In Section 3, the above analysis is extended to k-step block methods, obtaining a similar conclusion that there is only one of these methods which is the

* Corresponding author at: Scientific Computing Group, Universidad de Salamanca, Plaza de la Merced, 37008 Salamanca, Spain.
E-mail addresses: higra@usal.es (H. Ramos), paul_p_popescu@yahoo.com (P. Popescu).

http://dx.doi.org/10.1016/j.amc.2017.08.036

most efficient. Some numerical examples are considered in Section 4 to show the performance of the different formulations of block methods. Finally, some conclusions are outlined in Section 5.

2. Analysis of the 2-step block methods

The first appearance of a 2-step block method seems to have been in [9] attributed to B. Dimsdale and R. F. Clippinger, where the method was written in the form

$$\begin{cases} y_{n+1} - \frac{1}{2}y_{n+2} = \frac{1}{2}y_n + \frac{h}{4}f_n - \frac{h}{4}f_{n+2} \\ y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}). \end{cases} \tag{2}$$

This method was mentioned also in [5], and later in [2], where the two previous works were cited.

Later, Onumanyi et al. in [10] presented the 2-block method given by the two formulas

$$\begin{cases} y_{n+1} = y_n + \frac{h}{12}(5f_n + 8f_{n+1} - f_{n+2}) \\ y_{n+2} = y_{n+1} + \frac{h}{12}(-f_n + 8f_{n+1} + 5f_{n+2}). \end{cases} \tag{3}$$

This 2-step block method has also appeared in [11–13] and [14].

Finally, in [15] Hongjiong and Bailin presented the 2-step block method that follows

$$\begin{cases} y_{n+1} = y_n + \frac{h}{12}(5f_n + 8f_{n+1} - f_{n+2}) \\ y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}). \end{cases} \tag{4}$$

Although we have made a vast searching, it possibly might have some other 2-step block methods of this kind in literature. What is the difference between these methods? Are there more possibilities? Which of them is the most efficient? To answer these questions we are going to proceed in developing all the 2-step block methods which are similar in appearance to the ones presented above.

We consider the grid points given by $x_n, x_{n+1} = x_n + h, x_{n+2} = x_n + 2h$. For solving the problem in (1) on the interval $[x_n, x_{n+2}]$ we consider the approximation of its solution $y(x)$ by a polynomial $p(x)$ given by

$$y(x) \simeq p(x) = \sum_{j=0}^{3} a_j x^j, \tag{5}$$

where the $a_j \in \mathbb{R}$, are real unknown parameters to be determined. The usual way to determine the values of these parameters relies on imposing appropriate collocation conditions to $p(x)$ and $p'(x)$ at the points x_n, x_{n+1}, x_{n+2}. Choosing four equations of the set

$$\left\{p(x_n + ih) = y_{n+i}\,,\, p'(x_n + ih) = f_{n+i}\right\}, \quad i = 0, 1, 2$$

where as usually y_{n+i} and f_{n+i} are approximations for the solution and the derivative at the given points, $y_{n+i} \simeq y(x_n + ih)$, $f_{n+i} \simeq y'(x_n + ih) = f(x_n + ih, y(x_n + ih))$, we obtain a system of four algebraic equations in four unknowns (the a_j , $j = 0, 1, 2, 3$). After solving the above system we substitute the obtained values in the polynomial $p(x)$, and the remaining two equations after substituting the a_j will constitute the block method. All of the 2-step block methods shown before may be obtained in this way.

The collocation conditions are given explicitly by

$$\begin{aligned} &a_0 + x_n a_1 + x_n^2 a_2 + x_n^3 a_3 - y_n = 0 \\ &a_0 + x_{n+1} a_1 + x_{n+1}^2 a_2 + x_{n+1}^3 a_3 - y_{n+1} = 0 \\ &a_0 + x_{n+2} a_1 + x_{n+2}^2 a_2 + x_{n+2}^3 a_3 - y_{n+2} = 0 \\ &a_1 + 2x_n a_2 + 3x_n^2 a_3 - f_n = 0 \\ &a_1 + 2x_{n+1} a_2 + 3x_{n+1}^2 a_3 - f_{n+1} = 0 \\ &a_1 + 2x_{n+2} a_2 + 3x_{n+2}^2 a_3 - f_{n+2} = 0. \end{aligned}$$