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# The Kantorovich variant of an operator defined by D. D. Stancu ☆

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ABSTRACT

In this note we introduce Kantorovich variant of the operators considered by Stancu (1998) based on two nonnegative parameters. Here, we prove an approximation theorem with the help of Bohman–Korovkin's principle and find the estimate of the rate of convergence by means of modulus of smoothness and Lipschitz type function for these operators. In the last section of the paper, we show the convergence of the operators by illustrative graphics in Mathematica to certain functions.



## 1. Introduction

For $f \in C(I)$, with $I = [0, 1]$, the classical Bernstein polynomials are defined as follows:

$$\mathcal{B}_n(f; x) = \sum_{\nu=0}^{n} p_{n,\nu}(x) f\left(\frac{\nu}{n}\right),$$

where $p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu}(1-x)^{n-\nu}$ is the Bernstein basis.

Stancu [36] proposed the Bernstein type operators based on two parameters $r, s \in \mathbb{N} \cup \{0\}$, as follows:

$$(S_{n,r,s})f(x) = \sum_{\mu=0}^{n-sr} p_{n-sr,\mu}(x) \sum_{\nu=0}^{s} p_{s,\nu}(x) f\left(\frac{\mu+\nu r}{n}\right). \tag{1.1}$$

For the special case $r = s = 0$, these operators reduces to the operators $\mathcal{B}_n(f; x)$.

Razi [35] proposed a Bernstein–Kantorovich operators based on Pólya–Eggenberger distribution. He discussed the degree of approximation and the rate of convergence for these operators. Abel and Heilmann [1] studied the complete asymptotic expansion of the Bernstein–Durrmeyer operators. Gonska and Paltanea [26] introduced genuine Bernstein–Durrmeyer operators involving one parameter family of linear positive operators and gave the simultaneous approximation for these operators. In 2014, Cárdenas-Morales and Gupta [13] proposed a two-parameter family of Bernstein–Durrmeyer type operators based on Polya distribution and studied a Voronovskaja type asymptotic theorem. Abel et al. [2] introduced the Durrmeyer

☆ Dedicated to the memory of great Mathematician Prof. D. D. Stancu.

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type modification of the operators (1.1) defined by

$$\mathcal{S}_{n,r,s}(f;x)=\sum_{\mu=0}^{n-sr}p_{n-sr,\mu}(x)\sum_{\nu=0}^{s}p_{s,\nu}(x)(n+1)\int_0^1 p_{n,\mu+\nu r}(t)f(t)dt. \tag{1.2}$$

The authors derived a complete asymptotic expansion and studied some basic approximation theorems for these operators. Very recently, Gupta et al. [27] considered the Durrmeyer variant of the Baskakov operators involving inverse Pòlya–Eggenberger distribution and discussed the local and global approximation properties. Kantorovich type modification of several sequences of linear positive operators has been made and studied for their approximation behaviour [cf. [3–5,7,9,11,12,14–18,25,29,32,33] etc.].

Inspired by their work, for $f\in C(I)$ we consider the following Kantorovich type modification of the operators (1.1) as:

$$\mathcal{K}_{n,r,s}(f;x)=\sum_{\mu=0}^{n-sr}p_{n-sr,\mu}(x)\sum_{\nu=0}^{s}p_{s,\nu}(x)\int_0^1 f\left(\frac{\mu+\nu r+t}{n}\right)dt. \tag{1.3}$$

In this article, we prove the basic approximation theorem for the operators (1.3) by using Bohman–Korovkin's theorem. We also find the estimates of the rate of convergence involving modulus of continuity and Lipschitz type function. Then, we study $A-$statistical convergence for these operators using Korovkin type statistical approximation theorem. Lastly, we show the convergence of the operators by illustrative graphics in Mathematica to certain functions.

Let $e_i(x)=x^i, i=0,1,2,\ldots$

**Lemma 1.** *For the operators $\mathcal{K}_{n,r,s}(f;x)$, we have*

(i) $\mathcal{K}_{n,r,s}(e_0;x)=1;$

(ii) $\mathcal{K}_{n,r,s}(e_1;x)=x+\dfrac{1}{2n};$

(iii) $\mathcal{K}_{n,r,s}(e_2;x)=x^2+\dfrac{x(1-x)}{n}\left(1+\dfrac{sr(r-1)}{n}\right)+\dfrac{x}{n}+\dfrac{1}{3n^2};$

(iv) $\mathcal{K}_{n,r,s}(e_3;x)=x^3+\dfrac{3x(3-2x)}{2n}+\dfrac{x(7-9x)+6rsx^2(r-1)(1-x)+4x^3}{2n^2}+\dfrac{1-2rsx\Big((5+9x-4x^2)-3r(1-x^2)-2r^2(1-3x+8x^2)\Big)}{4n^3}.$

Let $e_i^x(t)=(t-x)^i, i=1,2$

**Lemma 2.** *For the operators $\mathcal{K}_{n,r,s}(f;x)$, we get*

(i) $\mathcal{K}_{n,r,s}(e_1^x(t);x)=\dfrac{1}{2n};$

(ii) $\mathcal{K}_{n,r,s}(e_2^x(t);x)=\dfrac{x(1-x)}{n}\left(1+\dfrac{sr(r-1)}{n}\right)+\dfrac{1}{3n^2}.$

**Proof.** The proof of this lemma easily follows on applying Lemma 1. Hence, the details are omitted. □

**Lemma 3.** *For $f\in C(I)$, we have*

$$\|\mathcal{K}_{n,r,s}(f;x)\|\le\|f\|.$$

**Proof.** Applying the definition (1.3) and Lemma 1, we get

$$\begin{aligned}\|\mathcal{K}_{n,r,s}(f;x)\| &\le \sum_{\mu=0}^{n-sr}p_{n-sr,\mu}(x)\sum_{\nu=0}^{s}p_{s,\nu}(x)\int_0^1\left|f\left(\frac{\mu+\nu r+t}{n}\right)\right|dt\\ &\le \|f\|\mathcal{K}_{n,r,s}(e_0;x)=\|f\|.\end{aligned}$$

□

## 2. Direct estimates

**Theorem 1.** *Let $f\in C(I)$. Then $\lim_{n\to\infty}\mathcal{K}_{n,r,s}(f;x)=f(x)$, uniformly in $I$.*

**Proof.** Since $\mathcal{K}_{n,r,s}(1;x)=1,\ \mathcal{K}_{n,r,s}(t;x)\to x,\ \mathcal{K}_{n,r,s}(t^2;x)\to x^2$ as $n\to\infty$, uniformly in $I$. By Bohman–Korovkin's theorem, it follows that $\mathcal{K}_{n,r,s}(f;x)$ converges to $f(x)$ uniformly on $I$. □