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Accretive Darboux growth along a space curve

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ABSTRACT

In this article, we give a mathematical framework to model the kinematics of the surface growth of objects such as some crustacean creatures. For this, a growth velocity in the direction of the Darboux vector field is defined at each point on a spatial generating curve. A local orthonormal frame (alternative moving frame) is added to each point of the generating curve and a velocity is given in terms of local coordinate directions to obtain a system of differential equations. Using the analytical solutions of this system, various surface examples, including some seashells are provided and the shapes of these surfaces are illustrated.

1. Introduction

Although the elastic deformation of the relation between the growth and stress is an important factor for soft body accretion, the hard body accretions, such as seashells, horns, bones, antlers, teeth, etc. are not likely to be deformed. Thus, the shapes of hard bodies emerge in a beautiful, mathematically elegant spherical and self-similar global structures. One element is the use of the growth velocity vectors for the mathematical modeling of this growth process. It is formed by the evolution of a generating curve which is characterized by increased growth, rate of formation and space orientation. The direction and speed of the deposition are defined by a vector field defined at each point on the generating curve. The formulation has general applicability, allowing for arbitrary growth rates without any assumption on the shape of the generating curve.

Mathematical modeling of the surface growth give a relationship between the fields biology, geometry, mathematics and art. Also, mathematical modeling of biological processes is widely used in the search for biomedical phenomena. Especially for this, it is very important to define the disease-causing cells by means of a suitable mathematical model. An important application of studies in this field is given in the field of cancer biology [7,8]. In [9,10], Ilert gives a sufficient and simplified concept to model seashells. Later, his second paper formulates the problem of seashell geometry entirely in three dimensional real space, presenting those equations of most use for practical digital computer simulations. Then in [11], Deborah et al. present a method for modeling seashells, suitable for image synthesis purposes. They combine a geometric description of shell shapes with an activator–inhibitor model of pigmentation patterns on shell surfaces. In [12], Noshita et al. develop a novel method for deriving microscopic growth rates from the macroscopic shapes of gastropod shells. It is shown that the growth of some living things, such as sea shells, can be explained by simple mathematical laws and given some related examples in [13–15].

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In [1], Skalak et al. show how to construct various biologically relevant structures with appropriate selection of the growth vector field. An important element in their work is a completely local explanation. Also, the growth velocities are defined in terms of a fixed coordinate system. Then, in [3,5] Moulton et al. formulate a model of accretive growth which is fully local, generally applicable and analytically tractable. The importance of creating surfaces in this way is that it does not depend on computer algorithms. Rather, the shapes of complex surfaces are obtained by analytical solutions that depend on a few parameters, which can lead to a qualitative description of surface growth. The goal here is not to examine a specific structure (e.g. a spiral shell), but to begin with simple structures and slowly go to a complex structure and to establish local rules and relationships related to growth mechanisms. By doing so, it was determined how the local growth rules are linked to the spherical geometry that can be associated with biology. The developed model can generally be applied to the surface development processes. In the previous studies, it is considered a component of the growth velocity in the binormal direction for a planar curve to evolve and generate a surface. However, there are many different geometric shapes in the nature. Therefore, it may be more appropriate to use a spatial curve instead of a planar curve in the modeling of some growth processes. Therefore, it is considered a space curve instead of a planar curve in the present work. For the evolution of a space curve and a surface formation, there must be a growth component in the direction of the Darboux vector. For this, the growth vector field is defined in terms of an alternative moving frame $\{N, C, W\}$ on the generating curve. This frame ideally describes the growth in the direction of the Darboux vector. In this manner, a different perspective is given to understand and model the underlying mechanisms in the growth process.

In Section 2, it is described the modeling of the Darboux growth process and formulated equations for this recognition. InSection 3, the conditions for the local velocity to keep the shape constant during the growth process are derived by taking a general helix (a curve of constant slope, see [6]). In Section 4, it is obtained the equations for the velocity vector components for an arbitrary space curve. Also, some examples in each section are given to show the calculation steps and illustrate the relationship between geometry and the local velocity field. In Section 4.1, it is mentioned the biological interpretations for some of the obtained surfaces.

2. Definition of the Darboux growth

In [3,5], the growth velocity field $q(s, t)$ is defined along a generating curve $r(s, 0)$. The authors consider the accretive growth in the direction of the binormal vector at every point on the generating curve. It describes the evolution of the generating curve and hence defines a surface $r(s, t)$. In this section, the accretive growth along the Darboux direction at every point on a non-planar generating curve is defined. For this, the alternative moving frame $\{N, C, W\}$ is considered given in [2,4].

Let $r(s, t)$ be a differentiable space curve in $\mathbb{R}^3$. Take the orthonormal frame $\{d_1, d_2, d_3\}$ attached to $r(s, t)$. Without loss of generality, choose d_3 as the unit normal vector defined as:

$$r' \equiv \partial_s r(s,t) = D\lambda = \lambda d_3.$$

Here, $Dv = v_1 d_1 + v_2 d_2 + v_3 d_3$ for every $v \in \mathbb{R}^3$.

The matrices $U = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & u_1 & 0 \end{bmatrix}$ and $W = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & w_1 & 0 \end{bmatrix}$ describe the rotation and angular velocity of the local basis on the generating curve, respectively. Then,

$$D' \equiv \partial_s D = DU$$
$$\dot{D} \equiv \partial_t D = DW.$$

If the generating curve and attached frame are known, then the growth velocity can be defined as:

$$q(s,t) = \partial_t r = \dot{r}(s,t) = D\bar{q}$$

where $\bar{q} = (q_1, q_2, q_3)$ is the growth velocity expressed in the local frame. Therefore, following six nonlinear first order partial differential equations for seven depended variables (called compatibility equations) are obtained:

$$q_1' + u_2 q_3 + u_3 q_2 = \lambda w_2 \tag{2.1}$$
$$q_2' + u_3 q_1 - u_1 q_3 = -\lambda w_1 \tag{2.2}$$
$$q_3' + u_1 q_2 - u_2 q_1 = \dot{\lambda} \tag{2.3}$$
$$\dot{u}_1 - w_1' = u_2 w_3 - w_2 u_3 \tag{2.4}$$
$$\dot{u}_2 - w_2' = u_3 w_1 - w_3 u_1 \tag{2.5}$$
$$\dot{u}_3 - w_3' = w_2 u_1 - u_2 w_1. \tag{2.6}$$

Integrating the equation $\partial_t r(s,t) = D\bar{q}$ gives the surface. Since the alternative frame $\{N, C, W\}$ is used, the matrix U is in the following form:

$$U = \begin{bmatrix} 0 & g & -f \\ -g & 0 & 0 \\ f & 0 & 0 \end{bmatrix}$$