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Conserved quantities for Hamiltonian systems on time scales

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ABSTRACT

Conserved quantities for Hamiltonian systems on time scales with nabla derivatives and delta derivatives are presented. First, Hamilton principle on time scales with nabla derivatives is established and Hamilton canonical equation with nabla derivatives is obtained. Second, Noether identity and Noether theorem for Hamiltonian systems with nabla derivatives are achieved. Third, Hamilton canonical equation with delta derivatives, Noether identity and Noether theorem for Hamiltonian systems with delta derivatives are gotten through duality principle on the basis of the corresponding results with nabla derivatives. Fourth, some special cases of Noether identity and Noether theorem are given. And finally, two examples are devoted to illustrate the methods and results.

1. Introduction

A time scale is an arbitrary nonempty closed subset of real numbers, this definition was introduced by Stefan Hilger [1] in 1988. It follows from the definition that differential equations and difference equations are the special cases of time scales. However, from the definition, one can get much more results since there are many other nonempty closed subset of real numbers than just the set of real numbers or integers. Therefore, unification and extension are two main features of time scale calculus. The contents in detail can be found in Refs. [2,3].

Time scale calculus mainly focuses on two kinds of calculus: time scale nabla calculus and time scale delta calculus. Atici and Guseinov [4] introduced the nabla calculus of variations. Bohner [5] and Hilscher and Zeidan [6] introduced the delta calculus of variations, and established the Euler–Lagrange equations. Ferreira and Torres [7] studied high-order calculus of variations on time scales and obtained high-order Euler–Lagrange equations with delta derivatives. And the corresponding results with nabla derivatives were presented by Martins and Torres [8]. The theories of both delta and nabla calculus of variations are very important. For example, the nabla calculus has been used to solve the extreme problems in economics [4,9].

In 1918, Noether [10] introduced a method to find the solution of differential equations. Since then, this method was extended [11–18]. Recently, Noether theorems on time scales, including delta calculus and nabla calculus, have come out [19-24]. Bartosiewicz and Torres [19] first obtained the Noether theorem for holonomic and conservative systems on time scales with delta derivatives. Malinowska and Martins [20] established the second Noether theorem on time scales with delta derivatives. Martins and Torres [21] obtained Noether-type theorem with nabla derivatives. Malinowska and Ammi [22] presented the Noether theorem for control problem on time scales with delta derivatives. Cai et al. [23] studied Noether

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identity and conserved quantity for nonconservative and nonholonomic systems on time scales with delta derivatives. Song and Zhang [24] obtained Noether theorem for Birkhoffian systems on time scales with delta derivatives, and so on.

If one already have proven some results with delta derivatives (resp. nabla), then the results with nabla (resp. delta) derivatives can be obtained from the similar steps. In order to find a direct method to get results for nabla (resp. delta) calculus, Caputo [25] introduced duality principle, which can avoid reproving the results for nabla (resp. delta) calculus if we already have proven for the delta (resp. nabla) case. Duality principle is a simple and effective approach, and through which some important results have been achieved [21,25]. For example, Caputo [25] proved integration by parts and Euler–Lagrange equation with nabla derivatives. Martins and Torres [21] gave Noether identity, Noether theorem and a DuBois–Reymond necessary optimality condition with nabla derivatives.

In this paper, we intend to establish Noether theorems for Hamiltonian systems on time scales, including both delta derivatives and nabla derivatives. Using Hamilton principle, we obtain some results with nabla derivatives. Then based on those results, we get the corresponding results with delta derivatives. The paper is organized as follows. First, some basic knowledge about time scales and duality principle is reviewed. Second, Hamilton canonical equation, Noether identity and Noether theorem on time scales with nabla derivatives are presented. Third, Hamilton canonical equation, Noether identity and Noether theorem for Hamiltonian systems on time scales with delta derivatives are gotten through duality principle. Fourth, some special cases are discussed. And finally, two examples are showed to illustrate the method and results.

2. Preliminaries

First, we review some knowledge about time scales [2,3].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. It follows from the definition that the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ are special cases of $\mathbb{T}$.

Definition 1. Let $\mathbb{T}$ be a time scale,

(1) The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined as follows

$$\sigma(t) = \inf\{x \in \mathbb{T} : x > t\}, \quad \rho(t) = \sup\{x \in \mathbb{T} : x < t\}, \quad t \in \mathbb{T},$$

where $\inf\emptyset = \sup\mathbb{T}$, $\sup\emptyset = \inf\mathbb{T}$.

(2) The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ and the backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ are defined as $\mu(t)=\sigma(t)-t$, $\nu(t)=t-\rho(t)$, $t \in \mathbb{T}$.

(3) t is called right-scattered, left-scattered, isolated, right-dense, left-dense and dense if $\sigma(t) > t$, $\rho(t) < t$, $\rho(t) < t < \sigma(t)$, $\sigma(t)=t$, $\rho(t) < t < \sigma(t)$, $\sigma(t)=t$, $\rho(t)=t$ and $t=\sigma(t)=\rho(t)$ hold respectively.

(4) $\mathbb{T}_k = \mathbb{T}\backslash[\inf\mathbb{T}, \sigma(\inf\mathbb{T}))$, $\mathbb{T}^k = \mathbb{T}\backslash(\sup\mathbb{T}, \rho(\sup\mathbb{T})]$ if $\inf\mathbb{T} > -\infty$, $\sup\mathbb{T} < \infty$; $\mathbb{T}_k = \mathbb{T}^k = \mathbb{T}$ if $\inf\mathbb{T} = -\infty$, $\sup\mathbb{T} = +\infty$.

Definition 2. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, $t \in \mathbb{T}_k$ (resp. $t \in \mathbb{T}^k$), then $\forall\varepsilon > 0$, $\exists\pi > 0$ such that

$$\left|f(\rho(t)) - f(\omega) - f^{\nabla}(t)[\rho(t) - \omega]\right| \le \varepsilon|\rho(t) - \omega|$$

(resp. $|f(\sigma(t)) - f(\omega) - f^{\Delta}(t)[\sigma(t) - \omega]| \le \varepsilon|\sigma(t) - \omega|$)
for all $\omega \in (t - \pi, t + \pi) \cap \mathbb{T}$. $f^{\nabla}(t)$ (resp. $f^{\Delta}(t)$) is called nabla (resp. delta) derivative at t. If $f^{\nabla}(t)$ (resp. $f^{\Delta}(t)$) exists for all $t \in \mathbb{T}_k$ (resp. $t \in \mathbb{T}^k$), then $f^{\nabla}(t)$ (resp. $f^{\Delta}(t)$) is called nabla (resp. delta) differentiable on $\mathbb{T}_k$ (resp. $\mathbb{T}^k$).

Definition 3. 1) f is ld-continuously nabla differentiable (resp. rd-continuously delta differentiable), denoting $f \in C^1_{\text{ld}}$, (resp. $f \in C^1_{\text{rd}}$) if $f^{\nabla}(t)$ (resp. $f^{\Delta}(t)$) exists for all $t \in \mathbb{T}_k$ (resp. $t \in \mathbb{T}^k$) and $f^{\nabla} \in C_{\text{ld}}$ (resp. $f^{\Delta} \in C_{\text{rd}}$).

2) $F : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of f provided $F^{\nabla}(t)=f(t)$ holds for all $t \in \mathbb{T}_k$. Furthermore, Every ld-continuous function has a nabla antiderivative.

Some other useful properties:

$$f \circ \rho = f^{\rho} = f - \nu(t)f^{\nabla} \quad \left(\text{resp.} \quad f \circ \sigma = f^{\sigma} = f + \mu(t)f^{\Delta}\right).$$

$$(\alpha f + \beta g)^{\nabla}(t) = \alpha f^{\nabla}(t) + \beta g^{\nabla}(t) \quad \left(\text{resp.} \quad (\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t)\right).$$

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g^{\rho}(t) + f(t)g^{\nabla}(t) = f^{\rho}(t)g^{\nabla}(t) + f^{\nabla}(t)g(t)$$

$$\left(\text{resp.} \quad (fg)^{\Delta}(t) = f^{\Delta}(t)g^{\sigma}(t) + f(t)g^{\Delta}(t) = f^{\sigma}(t)g^{\Delta}(t) + f^{\Delta}(t)g(t)\right).$$

Lemma 1 [8]**.** *Let $f \in C([a, b], \mathbb{R})$, if*

$$\int_a^b f(t)\eta^{\nabla}(t)\nabla t = 0 \quad \textit{for all} \quad \eta \in C^1([a, b], \mathbb{R}) \quad \textit{with} \quad \eta(a) = \eta(b) = 0,$$

then

$f(t)=c$ $\forall t \in [a, b]_k$ for some $c \in \mathbb{R}$.

Then we review some basic knowledge about the duality principle [25].