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# An asymptotic expansion for a class of biorthogonal polynomials with respect to a measure on the unit circle

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ABSTRACT

We consider the system of biorthogonal polynomials $\{P_n, Q_n\}_{n \geq 0}$ with respect to a complex valued measure supported on the unit circle and give a uniform compound asymptotic expansion formula consisting of the sum of two inverse factorial series, giving the explicit expression of the terms and including error bounds. As a consequence we prove that the set of accumulation points of the zeros these polynomials is included in the unit circle. Some numerical experiments are included.



## 1. Introduction and statement of the results

In this paper, we present an asymptotic expansion for a system of biorthogonal polynomials introduced by Askey [2]. Motivated by the fact that the families of Hermite, Jacobi, and Laguerre orthogonal polynomials are orthogonal with respect to the normal, beta and gamma distributions, respectively, Askey found a complex measure with support on the unit circle that is of beta function type and pointed out that there is a family of biorthogonal polynomials for this measure. To be more precise, we are interested in the two-parameter system $\{P_n, Q_n\}_{n \geq 0}$ of polynomials given by

$$P_n(z;\alpha,\beta) = {}_2F_1(-n, \alpha+\beta+1; 2\alpha+1; 1-z)$$
$$Q_n(z;\alpha,\beta) = P_n(z;\alpha,-\beta), \tag{1}$$

which is biorthogonal with respect to the complex valued weight $\omega(\theta) = (1-e^{\imath\theta})^{\alpha+\beta}(1-e^{-\imath\theta})^{\alpha-\beta} = (2-2\cos\theta)^{\alpha}(-e^{\imath\theta})^{\beta}, \theta \in [-\pi,\pi], \Re(\alpha) > -\frac{1}{2}$, that is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(e^{\imath\theta};\alpha,\beta)Q_m(e^{-\imath\theta};\alpha,\beta)\omega(\theta)d\theta = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha-\beta+1)}\frac{n!}{(2\alpha+1)_n}\delta_{n,m}, \tag{2}$$

where $\Gamma$ denotes the Euler Gamma function.

The biorthogonality (2) was stated in [2] in a slightly different form and a formal proof was given in [3], see [34] for some historical remarks.

Asymptotic properties for Toeplitz and Hankel determinants, for a more general class of weights including $\omega$, i.e., weights with a fixed number of Fisher–Hartwig singularities [9,16,17] has been obtained by Basor, Tracy and also other authors in a series of papers. More recently, Deift et al. [7] by using the Riemann–Hilbert approach obtained the general non-degenerate

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asymptotic behavior for Toeplitz determinants for such weights, as conjectured by Basor and Tracy. They also obtained asymptotic expansions for Hankel and Toeplitz–Hankel determinants, see the references within this last paper. Basor's classic work [4], as Askey has already pointed out in [2], suggests that the biorthogonal system (1) gives the right analogue of the Jacobi polynomials on the unit circle rather than the polynomials given by Szegö [32, (11.5.4)]. Without having an explicit formula for the biorthogonal system polynomials, and by using an indirect method of solving the strong Szegö limit problem for Toeplitz determinants, she obtained an asymptotic formula for the biorthogonal system (1) analogous to the existent for Jacobi polynomials in terms of Bessel functions, c.f., [32, (8.21.17)]. Having the explicit expression of the biorthogonal system, Askey [2] obtains the formula

$$P_n(e^{\frac{\imath\theta}{n}};\alpha,\beta)\sim {}_1F_1(\alpha+\beta+1;2\alpha+1;\imath\theta),\quad \text{as}\quad n\to\infty, \tag{3}$$

which is analogous to the one for Jacobi polynomials $P_n^{(\alpha,\beta)}$

$$n^{-\alpha}P_n^{(\alpha,\beta)}(\cos\theta/n)\sim\left(\frac{\theta}{2}\right)^{-\alpha}J_\alpha(\theta),\quad \text{as}\quad n\to\infty.$$

Progress in understanding the asymptotic behavior on compact subsets of the complex plane of orthogonal polynomials for weights having Fisher–Hartwig singularities has been obtained by Deift et al. [7] by using the Riemann–Hilbert approach and for positive weights with such singularities but without jumps by Martinez-Finkhelshtein et al. [21]. We mention also the contribution [18] using the Riemann–Hilbet method, there the authors obtained an uniform asymptotic expansion in the whole complex plane for an important subclass of the weight function $\omega$. Other asymptotic expansions for the Jacobi polynomials polynomials with varying parameters using the Riemann–Hilbert approach can be found in [14,15,20], here it is worth to notice the relation

$$P_n(z;\alpha,\beta)=const\,P_n{}^{(2\alpha+1,-n-\alpha+\beta-1)}(2z-1).$$

Temme [34] using an integral representation found an infinite power series asymptotic expansion for the biorthogonal system (1). He proved that, for $z$ and $(\alpha,\beta)$ varying on compact subsets of $\mathbb{C}\setminus\{0\}$ and $\{(\alpha,\beta)\in\mathbb{C}^2:\Re(\alpha+\beta)>-1,\Re(\alpha-\beta)\geq 0\}$, respectively, it holds that

$$P_n(z;\alpha,\beta)\sim\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)}z^{\alpha-\beta-1}\left(\frac{\ln z}{z-1}\right)^{2\alpha}$$
$$\times\left(\varphi_0\sum_{k=0}^{\infty}\frac{A_k}{(n+1)^k}+\varphi_1\sum_{k=0}^{\infty}\frac{B_k}{(n+1)^k}\right),\quad \text{as}\,n\to\infty,$$

where $\varphi_0=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(2\alpha+1)}{}_1F_1(\alpha+\beta+1,2\alpha+1;(n+1)\ln z)$, $\varphi_1=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(2\alpha+2)}{}_1F_1(\alpha+\beta+2,2\alpha+2;(n+1)\ln z)$ and $A_k$, $B_k$ are coefficients, depending on $z$, and defined by the relations [34, (2.13)]. Moreover, a bound for the remainder $R_p$ for this asymptotic expansion defined as

$$P_n(z;\alpha,\beta)=\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)}z^{\alpha-\beta-1}\left(\frac{\ln\,z}{z-1}\right)^{2\alpha}$$
$$\times\left(\varphi_0\sum_{k=0}^{p-1}\frac{A_k}{(n+1)^k}+\varphi_1\sum_{k=0}^{p-1}\frac{B_k}{(n+1)^k}+R_p\right),\ n,p\in\mathbb{N}, \tag{4}$$

is given by

$$|R_p|\leq\frac{M_p}{(n+1)^p}\left|\frac{\Gamma(\alpha+\beta+1)}{\Gamma(2\alpha+1)}\right||{}_1F_1(\alpha+\beta+1;2\alpha+1;(n+1)\Re\ln\,z)|,$$

where $M_p$ is some positive constant depending only on $p$. Temme remarked that the evaluation of the coefficients $A_k$ and $B_k$ is difficult, especially near or at unity. In particular, all the terms in the asymptotic formula (3) can be calculated using the recurrence formula for the coefficients $A_k$, $B_k$ by substituting $z=e^{\imath\frac{\theta}{n}}$ in (4). In [5], the authors gave an explicit expression for all the terms of formula (3) and an accurate bound for the remainder.

We mention also [13,25–27,30,31], there the authors using the integral representation approach obtained asymptotic expansions for the Gauss hypergeometric function

$${}_2F_1(a+e_1\lambda,b+e_2\lambda;c+e_3\lambda;z),$$

where $e_j=0,\pm1$, $j=1,2,3$ as $|\lambda|\to\infty$. These expansions hold for fixed values of $a$, $b$ and $c$, and is uniformly valid for $z$ properly restricted.

Asymptotic behavior of hypergeometric polynomials and of their zeros (or equivalently, of Jacobi polynomials, perhaps with varying parameters) is an active topic, more recent contributions can be found in [6,12,22,33].

In the present manuscript we give an uniform asymptotic expansion for the biorthogonal system (1) consisting of a sum of two inverse factorial series, for $z$ and $(\alpha,\beta)$ varying on compact subsets of $\mathbb{C}\setminus\{1\}$ and $\{(\alpha,\beta)\in\mathbb{C}^2:\Re(\alpha+\beta)>$