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Generalizations of Szőkefalvi Nagy and Chebyshev inequalities with applications in spectral graph theory

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ABSTRACT

Two weighted inequalities for real non-negative sequences are proven. The first one represents a generalization of the Szőkefalvi Nagy inequality for the variance, and the second a generalization of the discrete Chebyshev inequality for two real sequences. Then, the obtained inequalities are used to determine lower bounds for some degree-based topological indices of graphs.

1. Introduction

Let $a_1, a_2, \ldots, a_n$ be real numbers with the property $r \le a_i \le R, i = 1, 2, \ldots, n$. In 1918, Szőkefalvi Nagy [38] proved the following inequality

$$n\sum_{i=1}^{n} a_i^2 - \left(\sum_{i=1}^{n} a_i\right)^2 \ge \frac{n}{2}(R-r)^2. \tag{1}$$

This inequality is still attractive due to its numerous applications, such as in the study of relationships between various real numbers means, bounds of polynomial zeros, or eigenvalues of matrices (see [36,37,40]). In [31], an application of this inequality in spectral graph theory for determining lower bounds of some graph invariants was considered.

For sequences of non-negative real numbers $a = (a_i)$ and $b = (b_i)$, and a sequence $p = (p_i)$ of positive real numbers, the following inequality was proven in [32]:

$$\sum_{i=1}^{n} p_i a_i b_i - \frac{\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i}{\sum_{i=1}^{n} p_i} \ge \sum_{i=1}^{n-1} p_i a_i b_i - \frac{\sum_{i=1}^{n-1} p_i a_i \sum_{i=1}^{n-1} p_i b_i}{\sum_{i=1}^{n-1} p_i} \ge \cdots \ge \frac{p_1 p_2 (a_1 - a_2)(b_1 - b_2)}{p_1 + p_2} \ge 0 \tag{2}$$

where $a = (a_i)$ and $b = (b_i)$ are of the same monotonicity. Relation (2) is a generalization of Chebyshev's inequality.

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In this paper, we obtain inequalities that are stronger than (1) and (2). Then, we use them to establish lower bounds for some degree–based topological indices of graphs. By this, we improve a few earlier reported results.

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, $E=\{e_1,e_2,\dots,e_m\}$, be a simple graph with $n\ge 2$ vertices and m edges. Denote by $\Delta=d_1\ge d_2\ge\cdots\ge d_n=\delta>0$, $d_i=d(v_i)$, and $d(e_1)\ge d(e_2)\ge\cdots\ge d(e_m)\ge 0$ the sequences of its vertex and edge degrees, respectively. If two vertices v_i and v_j of the graph G are adjacent, we write $v_i\sim v_j$. Denote by $\mathbf{A}=(a_{ij})$ the adjacency matrix of G. The eigenvalues of $\mathbf{A}$, denoted by $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_n$, are referred to as the ordinary eigenvalues of the graph G. Some of their well known properties are [4]:

$$\sum_{i=1}^{n}\lambda_i=0 \quad\text{and}\quad \sum_{i=1}^{n}\lambda_i^2=\sum_{i=1}^{n}d_i=2m.$$

The largest eigenvalue λ_1 is known as the spectral radius of G.

The much investigated degree–based graph invariants, called the first and second Zagreb indices, M_1 and M_2, were defined in [17] and [16], respectively, as:

$$M_1=M_1(G)=\sum_{i=1}^{n}d_i^2=\sum_{v_i\sim v_j}(d_i+d_j)\quad\text{and}\quad M_2=M_2(G)=\sum_{v_i\sim v_j}d_i\,d_j.$$

In the same paper [17], an invariant F was encountered, defined as

$$F=F(G)=\sum_{i=1}^{n}d_i^3=\sum_{v_i\sim v_j}(d_i^2+d_j^2).$$

In the following 40 years, F escaped anybody's attention, and was therefore named "forgotten topological index" [12]. For more details of these topological indices see in [1,2,6,11,14,15,28,34].

By analogy with the first Zagreb index, by replacing vertex degrees by edge degrees, a so-called "reformulated first Zagreb index" EM_1 was defined as [27]

$$EM_1=EM_1(G)=\sum_{i=1}^{m}d(e_i)^2.$$

Evidently, $EM_1(G)=M_1(L(G))$, where $L(G)$ is the line graph of G.

A modified first Zagreb index ${}^mM_1$ was introduced in [27]:

$${}^mM_1={}^mM_1(G)=\sum_{i=1}^{n}\frac{1}{d_i^2},$$

which is a special case of the general zeroth–order Randić index Q_α, defined as (see for example [20,24,25]):

$$Q_\alpha=Q_\alpha(G)=\sum_{i=1}^{n}d_i^\alpha,$$

where α is an arbitrary real number. It is not difficult to observe that $Q_2=M_1,\ Q_3=F,\ Q_{-2}={}^mM_1$. For our paper, the case $\alpha=-1$ is also interesting [33]

$$ID=ID(G)=Q_{-1}=\sum_{i=1}^{n}\frac{1}{d_i}.$$

More on these and other degree–based topological indices can be found in [10,13–15,22,23,35,39,43] and in the references cited therein.

2. Generalizations of Szőkefalvi Nagy and Chebyshev inequalities

Let $I=\{1,2,\dots,n\}$, $J_k=\{i_1,i_2,\dots,i_k\}$, $J_k\subset I$, $1<i_1<i_2<\cdots<i_k<n$, $0\le k\le n-2$, $J_0=\emptyset$, be index sets, and $I_{n-k}=I-J_k$, whereby $I_n=I$, $I_2=\{1,n\}$ and $I_1=\{1\}$.

Let $a=(a_i)$ and $p=(p_i)$ be two non-negative sequences of real numbers. A weighted mean of order r, of a sequence $a=(a_i)$ with respect to $p=(p_i)$ is defined as

$$M_r(a,p;I)=\left(\frac{\sum_{i\in I}p_i\,a_i^r}{\sum_{i\in I}p_i}\right)^{1/r}.$$