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Chebyshev polynomials approach for numerically solving system of two-dimensional fractional PDEs and convergence analysis

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ABSTRACT

In this paper, an efficient numerical technique based on the Chebsyhev orthogonal polynomials is established to obtain the approximate solutions of system of two-dimensional fractional-order PDEs with initial conditions. We construct the corresponding differential operational matrix of fractional-order, and then transform the problem into a system of linear algebra equations. Compared with other analytical or semi-analytical methods, ours can achieve better convergence accuracy only small terms are expanded. Moreover the proposed algorithm is simple in theoretical derivation and numerical simulation. In our study, the convergence analysis of the system is emphatically investigated than other numerical approaches. Lastly, three numerical examples are applied to test the algorithm and that the obtained numerical results show that our approach is effective and robust.

1. Introduction

Fractional calculus theory is an important theoretical branch of mathematical theory [1], which has played an important role in various fields such as complex physical, mechanical, biological and engineering. For example, fractional calculus has been applied to model the nonlinear oscillation of earthquake [2], fluid-dynamic traffic [3], continuum and statistical mechanics [4], signal processing [5], control theory [6], and dynamics of interfaces between nanoparticles and subtracts [7]. In these practical applications, the fractional calculus has a certain geometric and physical meaning. In view of great practical significance for fractional calculus, so it is very important to study the fractional-order PDEs. In general, the analytical solutions of fractional-order PDEs cannot be easily obtained, so it is crucial to obtain the numerical solutions of these equations. In recent years, the researches on the numerical methods of fractional-order PDEs are increasingly growing, so as to approximately predict the tendency of the analytical solutions by the numerical solutions.

Various numerical approaches for different types of fractional-order PDEs have been presented. These methods include Chebyshev and Legendre polynomials methods [8,9], wavelets methods [10–12], piecewise constant orthogonal functions methods [13,14], spectral methods [15,16], collocation methods [17–19], differential transform methods [20,21], Adomian

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Decomposition Methods [22–24] and so on. In Ref. [25], the authors proposed Legendre wavelets method to solve system of nonlinear fractional differential equations. In Ref. [26], the authors acquired the numerical solution for system of fractional differential equations using a new approach called the Iterative Laplace transform method. In Ref. [27], the authors applied multiple fractional power series to obtain the analytical solution for system of nonlinear fractional Burger differential equations. In view of the above works, an orthogonal function based on the Chebyshev polynomials is applied to obtain the numerical solutions of system of two-dimensional fractional-order PDEs. This proposed algorithm will produce profound significance for solving real fractional problems.

The current paper is organized as follows: in Section 2, some basic definitions and mathematical preliminaries of fractional calculus are introduced. The operational matrix of fractional-order differentiation is given in Section 3. We mainly illustrate the proposed algorithm in Section 4. In Section 5, the convergence analysis of the system is investigated. In Section 6, the proposed approach is tested by three numerical examples. Finally, a conclusion is drawn in Section 7.

2. Preliminaries and notations

2.1. The basic definitions of fractional integral and differential operator

Definition 1. The Riemann–Liouville fractional integral operator ($I^\alpha f$) of order α is

$$(I^\alpha f)(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau, & \alpha>0;\\ f(t), & \alpha=0.\end{cases} \tag{1}$$

and the fractional differential operator ($D^\alpha f$) of order α is

$$(D^\alpha f)(t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\dfrac{\mathrm{d}^m}{\mathrm{d}t^m}\displaystyle\int_0^t \frac{f(s)}{(t-s)^{\alpha-m+1}}\mathrm{d}s, & \alpha>0, m-1\le\alpha<m;\\ \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha=m.\end{cases} \tag{2}$$

Definition 2. The Caputo definition of fractional differential operator (${}_cD^\alpha f$) of order α is defined as

$${}_cD^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau, & m-1\le\alpha<m;\\ \dfrac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}, & \alpha=m.\end{cases} \tag{3}$$

For the Caputo derivative, we have

$${}_cD^\alpha t^\beta=\begin{cases}0, & \text{for } \beta\in N_0 \text{ and } \beta<\lceil\alpha\rceil;\\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}t^{\beta-\alpha}, & \text{for } \beta\in N_0 \text{ and } \beta\ge\lceil\alpha\rceil \text{ or } \beta\notin N_0 \text{ and } \beta>\lfloor\alpha\rfloor.\end{cases} \tag{4}$$

2.2. Properties of the Chebyshev polynomials

The well-known Chebyshev polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula:

$$T_{i+1}(t)=2tT_i(t)-T_{i-1}(t),\quad i=1,2,\dots$$

where $T_0(t)=1$ and $T_1(t)=t$. In order to use these polynomials on the interval $x\in[0, 1]$, we define the Chebyshev polynomials by introducing the change of variable $t=2x-1$. Let the Chebyshev polynomials $T_i(2x-1)$ are denoted by $T_i(x)$, then $T_i(x)$ can be obtained as follows [28]:

$$T_{i+1}(x)=2(2x-1)T_i(x)-T_{i-1}(x),\quad i=1,2,\dots \tag{5}$$

where $T_0(x)=1$ and $T_1(x)=2x-1$. The analytic form of the Chebyshev polynomials $T_i(x)$ of degree i is given by

$$T_i(x)=i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!}x^k, \tag{6}$$

where $T_i(0)=(-1)^i$ and $T_i(1)=1$.

The orthogonally condition is

$$\int_0^1 T_j(x)T_k(x)w(x)\mathrm{d}x=h_k, \tag{7}$$

where $w(x)=\frac{1}{\sqrt{x-x^2}}$ and $h_k=\{\begin{smallmatrix}\frac{b_k}{2}\pi, k=j,\\ 0, k\neq j,\end{smallmatrix}\ b_0=2, b_k=1, k\ge1.$