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# Numerical computation of hypersingular integrals on the real semiaxis

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ABSTRACT

In this paper we propose some different strategies to approximate hypersingular integrals

$$\oint_0^{+\infty} \frac{\mathcal{G}(x)}{(x-t)^{p+1}}dx,$$

where $p$ is a positive integer, $t > 0$ and the integral is understood in the Hadamard finite part sense. Hadamard Finite Part integrals (shortly FP integrals), regarded as $p$th derivative of Cauchy principal value integrals, are of interest in the solution of hypersingular BIE, which model many different kind of Physical and Engineering problems (see [1] and the references therein, [2], [3, 4]).

The procedure we employ here is based on a simple tool like the "truncated" Gaussian rule (see [5]), conveniently modified to remove numerical cancellation. We will consider functions $\mathcal{G}$ having different decays at infinity. The method is shown to be numerically stable and convergent and some error estimates in suitable Zygmund-type spaces are proved. Finally, some numerical tests which confirm the efficiency of the proposed procedures are presented.



## 1. Introduction

Hypersingular integrals, defined in [6], are of interest, for instance, in the numerical solution of hypersingular integral equations. As it is known, such kind of equations are model for many physics and engineering problems (see [7] and the references therein, [2,3,7–9]).

There is a wide literature devoted to the computation of the Finite Part (FP) of divergent integrals

$$\oint_a^{+b} \frac{g(x)}{(x-t)^{p+1}}dx, \quad p \in \{1, 2, \dots\}, \quad a < t < b,$$

for bounded intervals $[a, b]$. Limiting ourselves to global approximation methods for $p > 0$, we mention among them [1,7,8,10–17,34]. An historical overview on the numerical methods for FP integrals and many properties holding in the case of bounded domains can be found in [1,7,8,18]. About the papers which employ Gauss-type rules, these are nearly all

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devoted to the interval $[-1,1]$. A more general approach introduced in [9] looks for determining a Gauss quadrature rule w.r.t the weight $\frac{w(x)}{(x-t)^2}$, where $w(x)$ is any Gauss classical weight on finite or infinite ranges. However, the Authors discuss computational details only in the interval $[-1,1]$ and for some choices of Jacobi weights $w$. So, FP integrals over unbounded domains received less attention in the past.

On the other hand hypersingular integrals

$$\oint_0^{+\infty} \frac{\mathcal{G}(x)}{(x-t)^{p+1}}dx, \quad p \in \{1,2,\ldots\}, t>0 \tag{1}$$

are employed in the solution of hypersingular integral equations coming from Neumann 2D elliptic problems on semiplanes by a Petrov-Galerkin infinite BEM approach [4]. In [4] FP integrals on $[a,+\infty)$, $a>0$ are reduced to the interval [0, 1] and approximated by means of product integration rules. Nevertheless, non linear transformations can get worse the density function $\mathcal{G}$ (see [19]), while the straightforward computation on unbounded ranges can add computational and also theoretical difficulties.

Thus, we propose here some global strategies to approximate integrals of the type (1). The proposed framework allows to consider functions $\mathcal{G}$ having different decays at infinity and uses different approaches, according to the position of $t>0$.

At first we consider the case $\mathcal{G}(x) = f(x)w_\alpha(x)$, where $w_\alpha(x) = e^{-x}x^\alpha$, $\alpha \ge 0$, is a Laguerre weight. Following a very standard way, we start from the decomposition

$$\begin{aligned}\mathcal{H}_p(fw_\alpha,t) &:= \oint_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx \\ &= \int_0^{+\infty} \frac{f(x)-\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}}w_\alpha(x)dx + \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\oint_0^{+\infty}\frac{w_\alpha(x)}{(x-t)^{p+1-k}}dx, \\ &=: \mathcal{F}_p(fw_\alpha,t) + \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\mathcal{H}_{p-k}(w_\alpha,t)\end{aligned} \tag{2}$$

focusing the attention on the first right-hand integral, since the remaining FP integrals are computable with high accuracy by standard routines (see Section 6). We use a simple tool like the Gauss–Laguerre rule properly modified in order to get a stable, convergent and efficient procedure to approximate the integral $\mathcal{F}_p(fw_\alpha,t)$. In particular, we use the "truncated" version of the Gauss–Laguerre rule [5] (see also [35]) in order to reduce the number of function computations and possible overflow ranges. Furthermore, for any fixed $t$, we select a suitable subsequence of "truncated" Gauss–Laguerre rule for avoiding the severe numerical cancellation arising when $t$ is "close" to a Gaussian node. The approach for $t$ "large" can be treated in a cheaper way, with a shrewd application of the Gaussian rule directly to $\mathcal{H}_p(fw_\alpha,t)$.

As second case we will consider density functions of the type $\mathcal{G}(x) = g(x)/(1+x)^\beta$, $\beta>1$. Indeed, by applying the aforesaid procedure to

$$\oint_0^{+\infty} \frac{\tilde{g}(x)}{(x-t)^{p+1}}e^{-x}dx, \quad \tilde{g}(x) = \frac{g(x)}{(1+x)^\beta}e^x, \tag{3}$$

the results may be rather poor, especially when $\mathcal{G}(x)$ "slowly" decays to zero as $x \to +\infty$ (see [20] about Gaussian rule deficiencies). For this reason, in some cases presented below, we show how to gain better results by making a preliminary change of variable and by applying then the above procedure. We complete this argument determining conditions on $g$ under which the global scheme is stable and fast convergent. Also in this case, when $t$ is "large" we suggest a different strategy.

Since the computation of the derivatives required for implementing the method can bring difficulties to the algorithm, we complete the description showing how to approximate $\{f^{(k)}\}_{k=0}^{p}$ by means of the derivatives of a suitable Lagrange polynomial interpolating $f$. In view of the behavior of the Lagrange polynomial sequence, under appropriate assumptions, the rate of convergence of the method remains unchanged, except the extra factor $\log m$.

The paper is organized as follows. In Section 2 some basic results about orthogonal polynomials and function spaces, needed to introduce the main results, are collected. Section 3 contains the definition of Hadamard finite part integrals over $(0,+\infty)$ for functions $f$ belonging to suitable Zygmund-type spaces, and some their properties. In Section 4 the numerical method to approximate $\mathcal{H}_p(fw_\alpha,t)$ is described and some results about the stability and the rate of convergence are stated. In the successive Section 5 we show how it is possible to speed up the convergence of the method for integrals of the type (3). In Section 6 we show how to avoid the computation of the derivatives of the function $f$. Section 7 contains some computational details useful in the implementation process. In Section 8 some numerical experiments are given to confirm the efficiency of the procedure. Moreover, comparisons with the method in [4] are shown. Finally, in Section 9 the proofs of the main results are stated.

## 2. Basic definitions and properties

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters