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Alternating direction numerical scheme for singularly perturbed 2D degenerate parabolic convection-diffusion problems

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ABSTRACT

In this article, we study the numerical solution of singularly perturbed 2D degenerate parabolic convection-diffusion problems on a rectangular domain. The solution of this problem exhibits parabolic boundary layers along $x = 0$, $y = 0$ and a corner layer in the neighborhood of (0, 0). First, we use an alternating direction implicit finite difference scheme to discretize the time derivative of the continuous problem on a uniform mesh in the temporal direction. Then, to discretize the spatial derivatives of the resulting time semidiscrete problems, we apply the upwind finite difference scheme on a piecewise-uniform Shishkin mesh. We derive error estimate for the proposed numerical scheme, which shows that the scheme is ε-uniformly convergent of almost first-order (up to a logarithmic factor) in space and first-order in time. Some numerical results have been carried out to validate the theoretical results.

1. Introduction

In this article, we consider the following singularly perturbed 2D degenerate parabolic convection-diffusion initial-boundary-value problem (IBVP) on a domain, $G = \Omega \times (0, T]$, $\Omega = (0, 1)^2$:

$$\begin{cases} L_\varepsilon u(x, y, t) - u_t(x, y, t) = f(x, y, t), & \text{in } G, \\ u(x, y, 0) = \phi_0(x, y), & (x, y) \in \Omega, \\ u(x, y, t) = 0, & (x, y, t) \in \partial\Omega \times (0, T], \end{cases} \tag{1.1}$$

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where

$$\begin{cases} L_\varepsilon u \equiv \varepsilon \triangle u + \mathbf{a}(x, y) \cdot \nabla u - b(x, y)u, \\ \mathbf{a} = (a_1, a_2), \ \ a_1(x, y) = \widehat{a}_1(x, y)x^p, \\ a_2(x, y) = \widehat{a}_2(x, y)y^q, \quad p, q \geq 1, \quad \forall (x, y) \in \overline{\Omega}, \\ \widehat{a}_1(x, y) \geq \alpha_1 > 0, \widehat{a}_2(x, y) \geq \alpha_2 > 0, \quad \forall (x, y) \in \overline{\Omega}, \\ b(x, y) \geq \beta > 0, \quad \forall (x, y) \in \overline{\Omega}. \end{cases}$$

We assume that the functions $\widehat{a}_1$, $\widehat{a}_2$, b, ϕ_0 are sufficiently smooth in $\overline{\Omega}$ and the source term f is sufficiently smooth in G. Further, we assume that ϕ_0 and f satisfy sufficient compatibility conditions at the corner points of the domain Ω and $0 < \varepsilon \ll 1$. Under these assumptions, the 2D parabolic IBVP (1.1) admits a unique solution, which exhibits parabolic boundary layers along $x = 0$, $y = 0$ and a corner layer in the neighborhood of (0, 0) (see [18]).

It is well-known that the solution of singular perturbation problems (SPPs) exhibits boundary layer(s), where the solution changes rapidly in the boundary layer region, and varies slowly and behaves smoothly in the outer region. Classical numerical methods may fail to yield satisfactory numerical approximate solution to these problems unless one reduces the step-size in comparison with the diffusion parameter. In order to obtain parameter-uniform numerical solutions to SPPs by classical finite difference schemes, one has to discretize the domain by layer-adapted nonuniform meshes. There are several methods available in the literature to obtain uniformly convergent numerical solution of SPPs, for more details, one can refer the books [7,11,18,19]. An efficient parallel boundary value technique is used in [15,21,23] to solve certain types of SPPs.

Singularly perturbed 2D steady state degenerate parabolic convection-diffusion problem arises in various branches of applied mathematics, including fluid dynamics, see [7]. Similarly, singularly perturbed degenerate 1D parabolic PDEs arise in the modeling of heat flow and mass transport near an oceanic rise (see [8]). Dunne et al. [5] described that degenerate PDEs can also be formed for the convection-diffusion without turning point problems posed on non-rectangular domain, specially when left and right boundaries are taken to be non-parallel straight lines. The multiple turning point problems arise in the modeling of thermal boundary layers in laminar flow (see [25]).

This paper is the first one, which analyzes an alternating direction finite difference scheme for singularly perturbed 2D degenerate parabolic PDEs (1.1). After applying this scheme for the time derivative, we obtain a set of 1D problems. Next, we apply the upwind finite difference scheme to discretize the spatial derivatives of those problems. We prove that the proposed method is ε–uniformly convergent of almost first-order in space and first-order in time.

Here, we provide a brief literature survey for degenerate problems. Vulanović and Farrell [25] studied the analytical and numerical solutions of multiple boundary turning point problem for ODEs. To obtain the numerical solution of singularly perturbed turning point problems exhibiting twin boundary layers, Natesan and Ramanujam proposed an initial-value technique in [13] and boundary-value technique in [14]. Natesan et al. obtained the numerical solution of singularly perturbed turning point problem in [12] by using classical finite difference scheme on piecewise-uniform Shishkin meshes, whereas they obtained the numerical solution of SPPs with weak boundary layer in [16]. Also, in [22], Vigo-Aguiar and Natesan solved SPPs exhibiting weak boundary layer, numerically by converting the second-order ODE into a system of two first-order ODEs. Ramos et al. used the non-standard algorithm on a piecewise uniform Shishkin mesh to solve nonlinear IVPs in [17].

Dunne et al. [6] applied the classical finite difference scheme on the piecewise-uniform Shishkin mesh to obtain the numerical solution of singularly perturbed 1D degenerate parabolic convection-diffusion problem and established almost first-order uniform convergence. Viscor and Stynes [24] solved singularly perturbed 1D degenerate parabolic problems numerically, by applying the classical finite difference scheme on Shishkin meshes. Clavero et al. [2] proposed a ε-uniformly convergent numerical scheme for singularly perturbed 1D degenerate convection-diffusion equation with a discontinuous source term, which exhibits an interior layer. Recently, Majumdar and Natesan [10] applied the Richardson extrapolation technique for solving singularly perturbed 1D degenerate convection-diffusion problems on Shishkin mesh.

In order to solve singularly perturbed 2D time-dependent parabolic convection– diffusion IBVPs, numerically, Clavero et al. [3] proposed a fractional-step method on the piecewise-uniform Shishkin mesh. Another alternating direction scheme was considered by Linß and Madden [9] for the singularly perturbed 2D parabolic reaction-diffusion IBVPs. Bujanda et al. [1] discussed the higher-order $\varepsilon-$uniform convergence scheme for the singularly perturbed reaction-diffusion problems.

The rest of the paper is arranged in the following way: In Section 2, we describe the semidiscrete problem by introducing an alternating direction scheme and study the uniform convergence of the semidiscrete scheme. In Section 3, we discretize the spatial domain using the piecewise uniform Shishkin mesh and then we apply the upwind finite difference scheme to approximate the semidiscrete problem. Section 4 contains several parts, first we study the asymptotic behavior of the semidiscrete problem, which follows the ε-uniform error estimate for the fully discrete scheme. In Section 5, we provide some numerical results to corroborate the theoretical estimates. The paper ends with conclusions.

Throughout this article, C denotes a generic positive constant, which is independent of the perturbation parameter ε, N, M and the mesh, where N, M are the number of sub-intervals in the spatial and temporal directions, respectively. Note that C