Applied Mathematics and Computation 313 (2017) 474–497

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Adaptive RBF-FD method for elliptic problems with point singularities in 2D☆

Dang Thi Oanh[a], Oleg Davydov[b,*], Hoang Xuan Phu[c]

[a] Division of Science-Technology and International Cooperation, University of Information and Communication Technology, Thai Nguyen University, Quyet Thang Commune, Thai Nguyen City, Vietnam
[b] Department of Mathematics, University of Giessen, Arndtstrasse 2, Giessen 35392, Germany
[c] Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet Road, Hanoi, Vietnam

ARTICLE INFO

Keywords:
RBF-FD
Meshless methods
Generalized finite differences

ABSTRACT

We describe and test numerically an adaptive meshless generalized finite difference method based on radial basis functions that competes well with the finite element method on standard benchmark problems with reentrant corners of the boundary, sharp peaks and rapid oscillations in the neighborhood of an isolated point. This is achieved thanks to significant improvements introduced into the earlier algorithms of Davydov and Oanh (2011), including a new error indicator of Zienkiewicz–Zhu type.

1. Introduction

Let us consider the Dirichlet boundary value problem: find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$Lu = f \quad \text{on}\, \Omega, \quad u|_{\partial\Omega} = g, \tag{1}$$

where L is a linear elliptic differential operator of second order, $\Omega \subset \mathbb{R}^2$ is a given bounded domain, the function f is defined on Ω, and the function g is defined on the boundary $\partial\Omega$ of Ω. A *generalized finite difference* discretization of the Dirichlet problem (1) is given by the following linear system with respect to the vector $\hat{u} = [\hat{u}_\xi]_{\xi\in\Xi}$

$$\sum_{\xi\in\Xi_\zeta} w_{\zeta,\xi}\hat{u}_\xi = f(\zeta), \quad \zeta \in \Xi_{\text{int}}; \qquad \hat{u}_\xi = g(\xi), \quad \xi \in \partial\Xi, \tag{2}$$

where

- $\Xi \subset \overline{\Omega}$ is the set of discretization centers;
- $\hat{u}$ represents the approximation of the solution u of (1) at the points $\xi \in \Xi$;

☆ This work was supported in part by Vietnam's National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01–2014.28, by a Natural Science Research Project of the Ministry of Education and Training under grant number B2015-TN06-02, and by the German Academic Exchange Service (DAAD) under the Programme "Research Stays for University Academics and Scientists."

* Corresponding author.
E-mail addresses: dtoanhtn@gmail.com (D.T. Oanh), ovdavydov@gmail.com, oleg.davydov@math.uni-giessen.de (O. Davydov), hxphu@math.ac.vn (H.X. Phu).

http://dx.doi.org/10.1016/j.amc.2017.06.006
0096-3003/

- $\partial\Xi := \Xi \cap \partial\Omega$ is the set of boundary discretization centers;
- $\Xi_{\text{int}} := \Xi \backslash \partial\Xi$ is the set of interior discretization centers;
- Ξ_ζ is a set (called the *stencil support* of ζ) that consists of the considered center ζ and some selected neighbor points $\xi_i \in \Xi$;
- $w_{\zeta,\xi} \in \mathbb{R}$ are the *stencil weights* chosen such that $\sum_{\xi\in\Xi_\zeta} w_{\zeta,\xi} u(\xi)$ is an approximation of $Lu(\zeta)$.

To set up the system, three tasks have to be addressed: (a) how to generate Ξ, (b) how to choose the stencil supports Ξ_ζ, and (c) how to compute suitable weights $w_{\zeta,\xi}$.

In the RBF-FD method the weights $w_{\zeta,\xi}$, $\xi \in \Xi_\zeta$, are generated through the interpolation with radial basis functions. Referring to [3,6] for further details and references, we briefly describe this approach. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a positive definite radial basis function [2], for example the Gaussian function

$$\phi(r) = e^{-\varepsilon^2 r^2}, \tag{3}$$

where ε is the *shape parameter*. Given $\zeta \in \Xi_{\text{int}}$ and $\Xi_\zeta = \{\zeta_0, \zeta_1, \ldots, \zeta_k\} \subset \Xi$, with $\zeta_0 = \zeta$, we set $\varphi_i(x) = \phi(\|x - \zeta_i\|)$, $x \in \mathbb{R}^2$, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. Assuming for simplicity that the operator L has the form

$$Lu(x) = \Delta u(x) + c(x)u(x),$$

we first find the weights w_i such that

$$\Delta s(\zeta) = \sum_{i=0}^{k} w_i u(\zeta_i),$$

where $s(x) := \sum_{i=0}^{k} a_i \varphi_i(x)$, $a_i \in \mathbb{R}$, satisfies the interpolation condition $s(\zeta_i) = u(\zeta_i)$, $i = 0, \ldots, k$. The vector $w = [w_i]_{i=0}^k$ can be computed by solving the linear system

$$\Phi_{\Xi_\zeta} w = [\Delta\varphi_i(\zeta)]_{i=0}^k, \quad \text{with} \quad \Phi_{\Xi_\zeta} := [\varphi_j(\zeta_i)]_{i,j=0}^k. \tag{4}$$

In particular, for the Gaussian , we have

$$\Phi_{\Xi_\zeta} = [e^{-\varepsilon^2\|\zeta_i-\zeta_j\|^2}]_{i,j=0}^k, \quad \Delta\varphi_i(\zeta) = 4\varepsilon^2 e^{-\varepsilon^2\|\zeta-\zeta_i\|^2}(\varepsilon^2\|\zeta - \zeta_i\|^2 - 1). \tag{5}$$

Assuming that the interpolant s provides a good approximation of the function u, we expect that $\Delta u(\zeta) \approx \Delta s(\zeta)$, and thus

$$Lu(\zeta) \approx \sum_{i=0}^{k} w_i u(\zeta_i) + c(\zeta)u(\zeta).$$

Therefore, the weights $w_{\zeta,\xi}$ in (2) are chosen as follows:

$$w_{\zeta,\zeta} = w_0 + c(\zeta), \qquad w_{\zeta,\zeta_i} = w_i, \quad i = 1, \ldots, k.$$

We refer to [5] for the bounds for the numerical differentiation error

$$\left| Lu(\zeta) - \sum_{\xi\in\Xi_\zeta} w_{\zeta,\xi} u(\xi) \right|.$$

The set of discretization centers Ξ does not have to form a grid or mesh, therefore RBF-FD is a meshless method [9]. For more complicated problems it is advantageous to adapt the distribution of the centers to the features of the domain Ω and/or to the singularities of the solution u. This can be achieved through *adaptive refinement* of Ξ. In [3], we suggested a refinement algorithm and an algorithm for stencil support selection, leading to an effective meshless method capable of competing with the finite element method on a number of benchmark test problems. However, further experiments have shown certain deterioration of the approximation quality after many refinement steps, and suboptimal performance for more difficult test problems.

This motivated the current study, where both stencil support selection and refinement have been improved. The new algorithms presented in Sections 2 and 3 deliver the stencil supports Ξ_ζ with more evenly distributed points, and the adaptively selected sets Ξ that better reflect the singularities of the solution. In the same time the improved method is more efficient because a costly post-processing step aimed at reducing the deterioration of the centers in the cause of subsequent refinements has been removed. One of the major differences in the refinement algorithm comparing to [3] is an error indicator of Zienkiewicz–Zhu type used instead of a simple gradient estimate. This leads to a significant improvement of the performance of the adaptive RBF-FD method for more difficult problems. Section 4 is devoted to numerical experiments with the test problems considered previously in [3], several test problems suggested in [8] as benchmarks for testing adaptive grid refinement, and a problem on a domain with a circular slit. In this paper we concentrate on the elliptic problems with *point singularities*, such as the reentrant corners of the boundary, sharp peaks and oscillations in the neighborhood of an isolated point. Problems with line or curve singularities, boundary layers and wave fronts require further adjustments of the