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A posteriori error estimates of finite element method for the time-dependent Navier–Stokes equations☆

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A B S T R A C T

In this paper, we consider the posteriori error estimates of Galerkin finite element method for the unsteady Navier–Stokes equations. By constructing the approximate Navier–Stokes reconstructions, the errors of velocity and pressure are split into two parts. For the estimates of time part, the energy method and other skills are used, for the estimates of spatial part, the well-developed theoretical analysis of posteriori error estimates for the elliptic problem can be adopted. More important, the error estimates of time part can be controlled by the estimates of spatial part. As a consequence, the posteriori error estimates in $L^\infty(0,T;L^2(\Omega)), L^\infty(0,T;H^1(\Omega))$ and $L^2(0,T;L^2(\Omega))$ norms for velocity and pressure are derived in both spatial discrete and time-space fully discrete schemes.

1. Introduction

Let $\Omega \in \mathbb{R}^2$ be a bounded domain, its boundary is assumed to satisfy a further condition **(A1)** stated below. We consider the posteriori error estimates of finite element method (FEM) for the following time-dependent Navier–Stokes equations

$$\begin{cases} u_t - \nu\Delta u + \nabla p + (u\cdot\nabla)u = f & \text{in } \Omega\times(0,T], \\ \text{div } u = 0 & \text{in } \Omega\times(0,T], \\ u = 0 & \text{on } \partial\Omega\times(0,T], \\ u = u_0 & \text{on } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where $u(x,t)$ is the velocity, $p = p(x,t)$ denotes the pressure, $f = f(x,t)$ the prescribed body force, u_0 the initial velocity, $\nu > 0$ the viscosity, and $T > 0$ the finite time.

In the last decade years, there is a growing demand of designing and developing reliable and efficient space-time numerical algorithms for the evolution equations. Most of these methods are based on the posteriori error estimators due to the significant advantages of the adaptive method [1,3,4]. Although the theory of the posteriori error estimates of FEM for elliptic problems is well-developed [2,5,25], the theory for parabolic problems is less developed, and only a few results have been published [10,17,22,26,27]. For the posteriori error estimates of FEM for the time-dependent Navier–Stokes problem, some

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meaningful results have been obtained. For example, by reducing the estimation of the error in a steady Stokes problem and using the so-called postprocessed technique, [11] has established a posteriori error estimates of both the semidiscrete and fully discrete cases. Based on the residuals of the Navier–Stokes equations, Prudhomme and Oden have provided the error control of finite element approximations for the time-dependent Navier–Stokes in spatial discrete scheme in [23]. By using the continuous differentiable function and the restriction operators between different spaces, Verfurth has provided a way to derive the posteriori error estimates of FEM for the nonlinear parabolic problem in [26,27] and Stokes equations [6,28].

In this paper, we adopt the techniques developed in [9,13,19–21] to treat the time-dependent Navier–Stokes equations. Firstly, we split the errors of velocity and pressure into two parts, by introducing the appropriate linearized Navier–Stokes reconstructions (see Definitions 3.1 and 4.1), the error equations can be reduced to a general time-dependent problem with continuous variables. The energy method and the other techniques established for parabolic problem can be employed, and the corresponding optimal order error estimates are obtained. For the linearized Navier–Stokes reconstructions, we can treat them by using the well-developed theoretical results and the skills (such as the duality argument) of the elliptic problem. As a consequence, the posteriori error estimates of velocity and pressure are established. One important thing we need to point out is that the estimates of parabolic equations can be controlled by the estimates obtained from the linearized Navier–Stokes reconstructions. Compared with the previous literature, the novel ingredients of this paper lie in: (1) The theoretical analysis tools are different. We construct the linearized Navier–Stokes reconstructions to solve the time-dependent Navier–Stokes problem, as a consequence, our analysis becomes quite clear and straightforward and the heart of the matter is succinct. (2) Some novel estimates are established by using the techniques of the posteriori error estimates of elliptic problem, the energy method and the duality argument, which extend the existed results of [18–21,29] from linear problem to the nonlinear problem.

This rest of this paper is organized as follows. In Section 2, the finite element method and some basic results of the time-dependent Navier–Stokes equations are recalled. Sections 3 and 4 are devoted to present the posteriori error estimates of both spatial discrete and time-space fully discrete formulations. Finally, some conclusions are given in Section 5.

2. Preliminaries

Standard Sobolev spaces are used in this paper. The spaces $L^2(\Omega)^m$ $(m = 1, 2, 4)$ are endowed with the L^2-scalar product $(\cdot,\ \cdot)$ and L^2-norm $\|\cdot\|_0$. For the mathematical setting of problem (1.1), we denote

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad D(A) = H^2(\Omega)^2 \cap X, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q dx = 0\},$$

where $A = -\Delta$ is the Laplace operator. Set the closed subset V of X be given by

$$V = \{v \in X; \mathrm{div} v = 0\}.$$

Assumptions **(A1)** and **(A2)** below specify the regularity of the domain and data required in Sections 3 and 4 to obtain the posteriori error estimates for the time-dependent Navier–Stokes equations in both spatial discrete form and fully discrete formulation. **(A1)** is an assumption on the regularity of the boundary $\partial\Omega$, which is satisfied if $\partial\Omega \in C^2$. (1) we need some regularities about the exact solutions; (2)

(A1). (see [15,16]) For the prescribed $g \in L^2(\Omega)^2$, the solution $(v, q) \in X \times M$ of the Stokes equation $-\Delta v + \nabla q = g$ satisfies $(v, q) \in H^2(\Omega) \times H^1(\Omega)$ and

$$\|v\|_{H^2} + \|q\|_{H^1} \le \|g\|_0$$

The assumptions about the data needed for our theoretical results are collected as follows.

(A2). (see [15,16]) Assume $u_0 \in D(A)$ and $f, f_t, f_{tt} \in L^\infty(0, \infty; Y)$ to satisfy

$$\|u_0\|_{H^2} + \sup_{[0,\infty)} (\|f\|_0^2 + \|f_t\|_0^2 + \|f_{tt}\|_0) \le M_1,$$

where $M_i > 0 (i = 1, 2)$ is a constant, only depends on Ω.

In this paper, we try to establish the bound of error estimates in both L^2 and H^1 norms, dual argument is used and 2 order regularity of velocity is employed. So we present some requirements about the domain and its boundary.

Theorem 2.1. *(see [17,18]) Under the conditions **(A1)** and **(A2)**, assume the constant M_1 is sufficiently small, then the solution of problem (1.1) exists for all $0 < T \le \infty$ and satisfies*

$$\sup_{0<t\le T} \|\nabla u\|_0 \le M_2, \quad \sup_{0<t\le T} \|u\|_{H^2} \le \widetilde{K},$$

where the constant $\widetilde{K}$ depends only on the bounds M_1 for f, M_2 and the domain Ω.

The bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$ are defined by

$$a(u, v) = \nu(\nabla u, \nabla v), \quad d(v, q) = -(\nabla q, v) = (q, \mathrm{div} v),$$