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Centralizer's applications to the inverse along an element

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ABSTRACT

In this paper, we firstly prove that the absorption law for one-sided inverses along an element holds, and derive the absorption law for the inverse along an element. We then obtain the absorption law for the inverse along different elements. Also, we prove that a left inverse of a along d coincides with a right inverse of a along d, provided that they both exist. Then, the reverse order law and the existence criterion for the inverse along an element are given by centralizers in a ring. Finally, we characterize the Moore–Penrose inverse of a regular element by one-sided invertibilities in a ring with involution.

1. Introduction

It is well-known that $a^{-1}+b^{-1}=a^{-1}(a+b)b^{-1}$ for any invertible elements a and b in a ring. The equality above is known as the absorption law. In general, the absorption law for group inverses, Drazin inverses, Moore–Penrose inverses, {1,3}-inverses and {1,4}-inverses does not hold. So, several papers such as [2,4,7] devoted to the study of these aspects.

Recently, authors [17] introduced a type of generalized inverse called one-sided inverses along an element, which can been seen as a generalization of group inverses, Drazin inverses, Moore–Penrose inverses and the inverse along an element. It is natural to consider whether the absorption law for such types of generalized inverses holds.

In this paper, we prove that the absorption law for one-sided inverses along an element holds in a ring. As applications, the absorption law for the inverse along the same element, i.e., $a^{\|d}+b^{\|d}=a^{\|d}(a+b)b^{\|d}$, is obtained. We then apply this result to obtain the absorption law for the inverse along different elements. Also, we prove that a left inverse of a along d coincides with a right inverse of a along d, provided that they both exist. Then, the reverse order law for the inverse along an element is considered. Furthermore, we derive an existence criterion of the inverse along an element by centralizers in a ring. Finally, we characterize the Moore–Penrose inverse of a regular element in terms of one-sided invertibilities, improving the results in [10,18].

Throughout this paper, we assume that R is an associative ring with unity 1. Let us now recall some notions of generalized inverses. We say that $a\in R$ is (von Neumann) regular if there exists x in R such that $a=axa$. Such x is called an inner inverse or {1}-inverse of a, and will be denoted by a^-.

Following [3], an element $a\in R$ is said to be Drazin invertible if there exist an element $b\in R$ and a positive integer k such that the following conditions hold:

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(a) $ab = ba$, (b) $b^2a = b$, (c) $a^k = a^{k+1}b$.

The element b satisfying the conditions (a)–(c) above is unique if it exists, which is called the Drazin inverse of a and is denoted by a^D. The smallest positive integer k in condition (c) is called the Drazin index of a and is denoted by ind(a). We call a group invertible if it is Drazin invertible with ind(a) = 1.

Let $*$ be an involution on R satisfying $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x+y)^* = x^* + y^*$ for all $x, y \in R$. An element a in R (with involution $*$) is Moore–Penrose invertible [12] if there exists $b \in R$ satisfying the following equations

(i) $aba = a$, (ii) $bab = b$, (iii) $(ab)^* = ab$, (iv) $(ba)^* = ba$.

Any element b satisfying the equations above is called a Moore–Penrose inverse of a. If such a b exists, then it is unique and is denoted by $a^\dagger$. If x satisfies the equations (i) and (iii), then it is called a {1, 3}-inverse of a, and is denoted by $a^{(1,3)}$. If x satisfies the equations (i) and (iv), then it is called a {1, 4}-inverse of a, and is denoted by $a^{(1,4)}$. It is known that a is Moore–Penrose invertible if and only if it is both {1,3}-invertible and {1,4}-invertible. Moreover, $a^\dagger = a^{(1,4)}aa^{(1,3)}$. Indeed, let $x = a^{(1,4)}aa^{(1,3)}$. We can check that x is the Moore–Penrose inverse a by Penrose's equations.

Let $a, d \in R$. We say that a is left (resp., right) inverse along d [17] if there exists $b \in R$ such that $bad = d$ (resp., $dab = b$) and $b \in Rd$ (resp., $b \in dR$). Such b is called a left (resp., right) inverse of a along d. By $a_l^{\|d}$ and $a_r^{\|d}$ we denote a left and a right inverse of a along d, respectively. In particular, a is invertible along d [8] if there exists b such that $bad = d = dab$ and $b \in dR \cap Rd$. Such b is unique if it exists, and is denoted by $a^{\|d}$. It is known [17] that a is both left and right invertible along d if and only if it is invertible along d. More results on (one-sided) inverses along an element can be referred to [1,17–19].

2. Absorption laws for (one-sided) inverses along an element

The main goal of this section is to illustrate that the absorption law for (left, right) inverses along an element holds in a ring. We first begin with the following lemma.

Lemma 2.1. *Let $a, b, d \in R$. Then*

(i) If $a_l^{\|d}$ and $b_r^{\|d}$ exist, then $a_l^{\|d}ab_r^{\|d} = b_r^{\|d}$ and $a_l^{\|d}bb_r^{\|d} = a_l^{\|d}$.

(ii) If $a_r^{\|d}$ and $b_l^{\|d}$ exist, then $b_l^{\|d}ba_r^{\|d} = a_r^{\|d}$ and $b_l^{\|d}aa_r^{\|d} = b_l^{\|d}$.

Proof. (i) Note that $a_l^{\|d}$ can be written as the form xd for some $x \in R$. Also, there exists $y \in R$ such that $b_r^{\|d} = dy$. Hence, it follows that $a_l^{\|d}ab_r^{\|d} = a_l^{\|d}ady = dy = b_r^{\|d}$ and $a_l^{\|d}bb_r^{\|d} = xdbb_r^{\|d} = a_l^{\|d}$.

(ii) By the symmetry of a and b. □

Applying Lemma 2.1, we get the absorption law for one-sided inverses along an element.

Proposition 2.2. *Let $a, b, d \in R$. Then*

(i) If $a_l^{\|d}$ and $b_r^{\|d}$ exist, then $a_l^{\|d} + b_r^{\|d} = a_l^{\|d}(a+b)b_r^{\|d}$.

(ii) If $a_r^{\|d}$ and $b_l^{\|d}$ exist, then $a_r^{\|d} + b_l^{\|d} = b_l^{\|d}(a+b)a_r^{\|d}$.

Proof. (i) It follows from Lemma 2.1(i) that

$$a_l^{\|d}(a+b)b_r^{\|d} = a_l^{\|d}ab_r^{\|d} + a_l^{\|d}bb_r^{\|d} = a_l^{\|d} + b_r^{\|d}.$$

(ii) It is a direct check by Lemma 2.1(ii). □

We next illustrate that a left inverse of a along d coincides with a right inverse of a along d, provided that they both exist. Moreover, they are also equal to the inverse of a along d.

Proposition 2.3. *Let $a, d \in R$. If a is both left and right invertible along d, then $a_l^{\|d} = a^{\|d} = a_r^{\|d}$.*

Proof. Suppose that a is both left and right invertible along d. Then $a_l^{\|d} = xd$ and $a_r^{\|d} = dy$ for some $x, y \in R$. Hence, $a_l^{\|d} = xd = xdaa_r^{\|d} = a_l^{\|d}aa_r^{\|d}$.

Similarly, $a_r^{\|d} = dy = a_l^{\|d}ady = a_l^{\|d}aa_r^{\|d}$, and consequently, $a_l^{\|d} = a_r^{\|d}$.

By the definition of the inverse along an element, we conclude that $a^{\|d}$ exists and $a^{\|d} = a_l^{\|d} = a_r^{\|d}$. □

As a corollary of Proposition 2.2, we obtain the absorption law for the inverse along an element.

Corollary 2.4. *Let $a, b, d \in R$ and let $a^{\|d}$ and $b^{\|d}$ exist. Then $a^{\|d} + b^{\|d} = a^{\|d}(a+b)b^{\|d}$.*

From Corollary 2.4, we know that the absorption law for the inverse along the same element holds in a ring. It is natural to consider whether the absorption law for the inverse along different elements, i.e., $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a+b)b^{\|d_2}$ holds? In general, $a^{\|d_1} + b^{\|d_2} = a^{\|d_1}(a+b)b^{\|d_2}$ does not hold as the following example shows.