Applied Mathematics and Computation 315 (2017) 131–149

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An interval for the shape parameter in radial basis function approximation

Jafar Biazar*, Mohammad Hosami

Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran

ARTICLE INFO

Keywords:
Radial basis function
Shape parameter
Kansa's method
Pseudo-spectral method
Method of line
Eigenvalue stability

ABSTRACT

The radial basis functions (RBFs) depend on an auxiliary parameter, called the shape parameter. Great theoretical and numerical efforts have been made to find the relationship between the accuracy of the RBF-approximations and the value of the shape parameter. In many cases, the numerical approaches are based on minimization of an estimation of the error function, such as Rippa's approach and its modifications. These approaches determine a value for the shape parameter. In this paper, we propose a practical approach to determine an interval, instead of a value, without any minimization and estimation of error function. The idea is based on adding a loop on the shape parameter, but not to minimize an error norm. Suitable values of the shape parameter are determined by take into account the practical convergence behavior of the problem. The proposed method is applied to some illustrative examples, including one and two-dimensional interpolation and partial differential equations. The results of the method are compared with those of some other approaches to confirm the reliability of the method. Furthermore, numerical stability of the method of line respect to the shape parameter is considered for time-dependent PDEs. The results show that, the proposed method can be used as an independent method to find a shape parameter, and also as an alternative for investigating the validity of the values of the other methods.

1. Introduction

Radial basis function (RBF) method is one of the important tools for interpolation of scattered data, and solving partial differential equations. In recent years, researchers have incorporated RBF method in several schemes in many practical problems, in science and engineering [1–10]. The interest in RBF method has two principal reasons; meshfree property and spectral convergence rate. But the accuracy of approximate solution depends on various factors. Some of the most important ones are, type of the RBF, position of the center points, and the value of the shape parameter in RBF. In this paper, we focus on the shape parameter. This parameter is a free parameter for controlling the shape of the RBFs. In most of the literatures, the value of the shape parameter is chosen by trial and error. There are some ad-hoc criteria for the choice of the value of this parameter [11–15]. Furthermore, many efforts have been devoted to obtain an optimal shape parameter by theoretical error estimates including the shape parameter [16–21]. But, numerical experiments show that the best value of the shape parameter, obtained via a theoretically scheme, may not be numerically optimal. There are also some numerical methods for choosing the shape parameter by minimizing an error function. Rippa is one of the pioneers in this group [22]. Some of

* Corresponding author.
E-mail addresses: biazar@guilan.ac.ir (J. Biazar), mohammad.hosami@gmail.com (M. Hosami).

http://dx.doi.org/10.1016/j.amc.2017.07.047
0096-3003/

Table 1
Some important RBFs.

Infinitely Smooth RBFs			Piecewise Smooth RBFs	
Name	Classical formula	Redefined formula	Name	Formula
Gaussian (GA)	e^{-r^2/c^2}	$e^{-\varepsilon^2 r^2}$	Linear	r
Multiquadric (MQ)	$\sqrt{c^2+r^2}$	$\sqrt{1+\varepsilon^2 r^2}$	Cubic	r^3
Inverse Multiquadric (IMQ)	$1/\sqrt{c^2+r^2}$	$1/\sqrt{1+\varepsilon^2 r^2}$	Thin Plate Spline (TPS)	$r^2\log(r)$
Inverse Quadric (IQ)	$1/(c^2+r^2)$	$1/(1+\varepsilon^2 r^2)$		

Note: Throughout this Paper, we use c to represent the shape parameter when using RBFs in classical formula and ε when using redefined formulas.

the works in this topic are due to Fasshauer and Zhang [23], Schuerer [24], Roque and Ferreira [25]. In all of these works, an optimal value of the shape parameter is suggested, by optimizing an error function. Moreover, in recent years, some new approaches have been presented in literatures to stably calculate RBF-approximation for very small value of the shape parameter [26–34]. Each of these methods has its own strength and weakness.

Although many researches have been done to study and determine appropriate values for the shape parameter, but so far no one has introduced any interval for suitable values of the shape parameter. The main aim of this paper is to introduce an optimal interval, and show that the proposed approach is practically suitable for choosing a good shape parameter, in scattered data interpolation and solving partial differential equations which are two important problems in science and engineering.

The organization of the paper is as follows. Section 2 deals with an introduction to RBF-approximation method. In Section 3, some strategies for selecting an appropriate value of the shape parameter are reviewed briefly. Two of these strategies are applied for comparison of the results with those of the proposed interval, in interpolation problems. In Section 4, an innovative approach is introduced that results in an interval including suitable values of the shape parameter. The results of numerical experiments, in one and two dimensional interpolations, are presented in Section 5. In Section 6, two approaches for selecting a good shape parameter for PDEs are reviewed. Moreover, the proposed method, introduced in Section 4, is adapted for PDEs. Additively, numerical stability of the method of line, for two time-dependent PDEs, is considered.

2. Radial basis function approximation

In radial basis function approximation, an unknown continuous (univariate or multivariate) function is approximated by a linear combination of RBFs.

2.1. RBF-interpolation

Radial basis functions are real valued functions, which depend on the Euclidean distance from a point of a given data, say $\{\mathbf{x}_i\}_{i=1}^n$, which are called center nodes. To interpolate a given scattered data $\{(\mathbf{x}_i, u_i)\}_{i=1}^n$ by RBFs, the interpolant will be presented as a combination of n RBFs, i.e.

$$u(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j \varphi(r_j), \quad \mathbf{x} \in \mathbf{R}^n \tag{1}$$

where $r_j = ||\mathbf{x} - \mathbf{x}_j||_2$, and $\phi(r)$is RBF. The commonly used RBFs are summarized in Table 1. Considering interpolation conditions $u(\mathbf{x}_j) = \mathbf{f}_j (j = 1y, \ldots, n)$ results in a system of linear equations as follows

$$\mathbf{A}\alpha = \mathbf{f}, \tag{2}$$

where $(\mathbf{A})_{ij} = \phi(r_{ij})$ and $r_{ij} = ||\mathbf{x}_i - \mathbf{x}_j||_2$.By solving the linear system (2), the coefficients α_j are determined. The problem (2) will be well-posed if and only if the matrix**A**is nonsingular. It is known that it is guaranteed for MQ, IMQ, and Gaussian. Also it is true for Thin-plate spline (TPS) by some modification on the interpolant (1) by adding a polynomial term [17,35]. For sufficiently smooth functions, the infinitely smooth RBFs (such as MQ, IMQ, and Gaussian) lead to spectral convergence, while the piecewise smooth RBFs give an algebraic rate of convergence [17,35]. This is a good incentive to use infinitely smooth RBFs. All these type of RBFs have a parameter (ε or c) which is called shape parameter. The accuracy of the RBF approximation depends heavily on the appropriate choice of the shape parameter. We concentrate on this topic in next sections.

2.2. RBF-approximation for partial differential equation

Radial basis function approximation can be also used as a reliable tool for solving PDEs. We briefly review a well-known RBF-approximation method for steady PDEs, known as Kansa's method [36,37].