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Higher-order derivative-free families of Chebyshev–Halley type methods with or without memory for solving nonlinear equations

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ABSTRACT

In this paper, we present two new derivative-free families of Chebyshev–Halley type methods for solving nonlinear equations numerically. Both families require only three and four functional evaluations to achieve optimal fourth and eighth orders of convergence. Furthermore, accelerations of convergence speed are attained by suitable variation of a free parameter in each iterative step. The self-accelerating parameter is estimated from the current and previous iteration. This self-accelerating parameter is calculated using Newton's interpolation polynomial of third and fourth degrees. Consequently, the *R*-orders of convergence are increased from 4 to 6 and 8 to 12, respectively, without any additional functional evaluation. The results require high-order derivatives reaching up to the eighth derivative. That is why we also present an alternative approach using only the first or at most the fourth derivative. We also obtain the radius of convergence and computable error bounds on the distances involved. Numerical experiments and the comparison of the existing robust methods are included to confirm the theoretical results and high computational efficiency. In particular, we consider a concrete variety of real life problems coming from different disciplines e.g., Kepler's equation of motion, Planck's radiation law problem, fractional conversion in a chemical reactor, the trajectory of an electron in the air gap between two parallel plates, Van der Waal's equation which explains the behavior of a real gas by introducing in the ideal gas equations, in order to check the applicability and effectiveness of our proposed methods.

1. Introduction

Finding rapidly and accurately the zeros of nonlinear functions is an interesting and challenging problem in the field of computational mathematics. In this study, we consider iterative methods for solving a nonlinear equation of the form $f(x) = 0$, where f: $D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a scalar function defined on an open interval D. Analytical methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical

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methods based on iterative procedure. One of the most famous and basic tool for solving such equations is the Newton's method [1,9,23] given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2, \ldots$. It converges quadratically for simple roots and linearly for multiple roots. Researchers have also proposed cubically convergent one-point methods. One such well-known scheme is the classical cubically convergent Chebyshev–Halley family [2] defined by

$$x_{n+1} = x_n - \left[1 + \frac{1}{2}\left(\frac{L_f(x_n)}{1 - \alpha L_f(x_n)}\right)\right]\frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \tag{1.1}$$

where $L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'(x_n)^2}$. This family includes Chebyshev's method for ($\alpha = 0$), Halley's method for ($\alpha = \frac{1}{2}$) and super-Halley method for ($\alpha = 1$). For more details, one may consult the references [3,4]. It is quite clear that the practical implementation of Chebyshev–Halley type methods is restricted in the problems where second derivative is difficult to evaluate. This fact has motivated many researchers to explore second derivative free two-point variants of Chebyshev–Halley methods, see [5,8] and references therein. However, Traub [9] proved that construction of any one-point iterative method without memory of order p, requires atleast $p - 1$ order derivatives. Due to these restrictions, the researchers are showing interest in the past years towards the development of multipoint methods for solving nonlinear equations.

Generally, multipoint iterative methods [24–26] are divided into two categories: with memory and without memory methods. According to the Kung–Traub conjecture [10], the order of convergence of any multipoint method without memory requiring d functional evaluations per iteration, cannot exceed the bound 2^{d-1}, called the optimal order. Also, efficiency of an iterative method is measured by the efficiency index [11] defined as $E = p^{\frac{1}{d}}$, where p is the order of convergence and d is the number of functional evaluations required per step. An extensive survey of multipoint methods can be found in the excellent book by Petković et al. [23]. The iterative methods for scalar nonlinear equations also have application in the construction of iterative methods to find generalized inverses. For instance, the Newton method has a close connection with quadratically convergent Schulz iterative method [6] for finding the Moore–Penrose inverse that can be written as $X_{n+1} = X_n - (2I - AX_n)$, where A is a matrix. One can easily obtain this iteration scheme by considering $f(X) = A - X^{-1}$ and then applying Newton's method. Recently, many researchers got attention to develop matrix iterative methods to find the generalized inverses, see [7] and references cited therein. For instance, a high-order stable numerical method for matrix inversion is presented in [12].

On the other hand, the basic idea for the construction of multipoint methods with memory was introduced by Traub [9]. He improved a Steffensen-like method by the reuse of information from the previous iteration using secant approach. In fact, he proposed the following method with memory:

$$\begin{cases} \gamma_0 \text{ is given}, \ \gamma_n = \dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n \geq 1, \\ x_{n+1} = x_n - \dfrac{\gamma_n f(x_n)^2}{f(x_n + \gamma_n f(x_n)) - f(x_n)}, \end{cases} \tag{1.2}$$

having R-order of convergence [14] atleast $1 + \sqrt{2} \approx 2.414$. A similar approach was applied to higher order multipoint methods in [13,15–17,21]. Surprisingly, this particular class of methods with memory is not completely dealt with in the literature in spite of their high computational efficiency.

The motive of this work is to present two new classes of derivative-free methods without memory based on Chebyshev–Halley family having optimal fourth and eighth orders of convergence. Each member of the proposed families supports Kung–Traub conjecture for $d = 3$ and $d = 4$, respectively. As a matter of fact, many higher-order derivative-free type methods without memory have been already derived in the literature using different techniques, see for instance [19,20,27] and the references cited therein. Hence, the proposed families can be regarded as an additional contribution to the subject but without an additional advantage. However, we do not have any higher-order derivative-free variants of Chebyshev–Halley type methods with memory till date.

With this aim, we further attempt to increase the convergence order of the proposed families by applying an accelerating procedure based on varying self-accelerating parameter calculated by Newton's interpolation polynomials in each iterative step. The R-orders of convergence of the proposed two-point and three-point derivative-free methods with memory are 6 and 12, respectively. As a result, efficiency indices increases from $E = 4^{\frac{1}{3}} \approx 1.587$ to $E = 6^{\frac{1}{3}} \approx 1.817$ and $E = 8^{\frac{1}{4}} \approx 1.682$ to $E = 12^{\frac{1}{4}} \approx 1.861$, which is much better than optimal three, four and five point methods without memory. It is noteworthy that the significant increase of convergence speed is achieved without additional functional evaluations. This means that the proposed methods with memory possess a very high computational efficiency, which is the main advantage of these methods in comparison to the methods without memory. It is found by way of illustrations that the proposed methods with and without memory are highly efficient in multi-precision computing environment.

2. Derivative-free two-point Chebyshev–Halley family and convergence analysis

In this section, we intend to develop a new derivative-free class of two-point Chebyshev–Halley type methods having optimal fourth-order convergence.