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Distributional study of finite-time ruin related problems for the classical risk model

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ABSTRACT

In this paper, we study some finite-time ruin related problems for the classical risk model. We demonstrate that some techniques recently developed in [37] and [6] can be used to study the joint distribution of the time of ruin and the maximum surplus prior to ruin, the joint distribution of the time of ruin and the maximum severity of ruin, and the distribution of the two-sided first exit time. Especially, by solving a Seal's type partial integro-differential equation we obtain an explicit (integral) expression for the finite-time Gerber–Shiu function, which is expressed in terms of the (infinite time) Gerber–Shiu function introduced in [12].

1. Introduction

Consider a classical risk process in which the insurer's surplus level at time $t(\geq 0)$, $U(t)$, is described as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

where $u(>0)$ is the initial surplus, $c(>0)$ is the constant premium rate per unit of time, and $\{X_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables representing the individual claim amounts, with common probability distribution function (d.f.) P, density function p, mean μ and Laplace transform (LT) $\hat{p}$. The counting process $\{N(t); t\geq 0\}$ denotes the number of claims up to time t and is assumed to be Poisson with parameter $\lambda(>0)$, and independent of $\{X_i\}_{i\geq 1}$. Furthermore, we assume that $c = (1+\theta)\lambda\mu$, where $\theta(>0)$ is the loading factor.

Define $T_u = \inf\{t \geq 0 : U(t) < 0\}$ to be the time of ruin, with $T_u = \infty$, if $U(t)\geq 0$ for all $t\geq 0$. Define $\psi(u,t) = \mathbb{P}(T_u \leq t) = 1 - \phi(u,t)$ to be the probability of ruin by (finite) time t with $\phi(u, t)$ being the corresponding finite-time survival probability; then $\psi(u,\infty) \triangleq \psi(u) = 1 - \phi(u) = \mathbb{P}(T_u < \infty)$ is the probability of ultimate ruin.

Let $S(t) = \sum_{i=1}^{N(t)} X_i$ be the aggregate amount of claims by time t and define $F(x, t)$ to be the distribution function of $S(t)$, i.e.,

$$F(x,t) = \mathbb{P}(S(t) \leq x) = e^{-\lambda t} + \sum_{n=1}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} P^{n*}(x), \quad x > 0, \; t \geq 0, \tag{1.1}$$

and denote $f(x,t) = \partial F(x,t)/\partial x$.

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The finite-time ruin related problems have been studied extensively for the classical risk model. Prahbu [33] and Seal [35] study the finite-time non-ruin (survival) probability; see also other references therein for early contributions and relevant results in classical risk theory. They show that the finite-time survival probability, $\phi(u, t)$, satisfies a partial integro-differential equation; through the Laplace inversion an explicit (integral) expression of $\phi(u, t)$ is obtained in terms of functions $F(x, t)$ and $f(x, t)$, as well as its (initial) expression at $u = 0$, which can be expressed as a function of $F(x, t)$ through integration.

For the continuous time classical risk model, Picard and Lefèvre [32] show that when the claim sizes are integer-valued, the distribution of the time of ruin T_u can be expressed in terms of generalized Appell polynomials, where T_u is viewed as the time of first crossing between a compound Poisson trajectory and an upper increasing boundary. The elegant work by Picard and Lefèvre [32] has stimulated further research in developing techniques and algorithms for the analytically or numerically evaluation of finite-time ruin probabilities. De Vylder [1] develops some techniques to numerically calculate finite-time ruin probabilities in the classical risk model based on the remarkable formula derived in [32], while De Vylder and Goovaerts [2] further derive general explicit analytic expressions for finite-time and infinite-time ruin probabilities assuming that the claim size distribution has a density on $[0, \infty)$, which are continuous versions of discrete ones obtained in [32]. Ignatov and Kaishev [13] derive explicit two-sided bounds for the finite-time non-ruin probability of an insurance company, where the premium income is described by an increasing function, the claim sizes are dependent and integer-valued random variables, and their inter-occurrence times are non-identically and exponentially distributed (parallel results for the same model when claim sizes are dependent and follow any continuous joint distribution can be found in [14]); see, also [15], for an improved version of the finite-time ruin probability formula of Ignatov and Kaishev [13] and numerical illustrations using *Mathematica*. Moreover, Rullière and Loisel [34] propose exact recursive formulas and a Seal-type formula to compute the finite-time ruin probability and compared the computational efficiency of their proposed formulas with Picard-Lefèvre and other existing formulas. Beside, Lefèvre and Loisel [20] generalize the formula derived in [32] for the finite-time ruin probabilities to the classical compound binomial and compound Poisson risk models.

By assuming that the arrival times within a fixed time interval satisfy an order statistic property, independent of claim amounts, Lefèvre and Picard [21] revisit the homogeneous risk model investigated by De Vylder and Goovaerts [2]; the distributional properties of the aggregate claim amount process are discussed and a closed-form expression for the finite-time non-ruin probability is derived, which is expressed in terms of a family of Appell polynomials. It is worth mentioning that Sendova and Zitikis [36] study some properties (including first two moments and related risk measures) of aggregate claims under the same claim arrival process with certain dependence structures between the claim sizes and the inter-claim times. Lefèvre and Picard [22] further consider two special cases of the model studied in [21] and an expression for the finite-time non-ruin probabilities is obtained by exploiting properties of an underlying family of Appell polynomials. See, also [23], for an interesting exploration of the mathematical relationship between the finite-time ruin probability in insurance and the final outcome distribution in epidemics.

Dickson and Willmot [10] find an expression for the probability density function of the time of ruin by inverting its Laplace transform via Lagrange's implicit function theorem, which provides an efficient way to study the finite-time ruin related quantities. The joint distribution of the time of ruin and some ruin related quantities are studied in [4,5] and [19]. The density of the time of ruin for the classical risk model with a constant dividend barrier can be found in [25]. The distribution of the number of claims in the two-sided first exit problem in the compound Poisson risk model is investigated in [26]. The finite-time ruin related problems for the Sparre Andersen models are further investigated in [8,18] and [9]. The finite-time ruin probability for a risk model with a Markovian arrival process (MAP) has been studied in [24].

Recently, Willmot [37] generalizes the approach presented in [33] and [35] and studies the joint distribution of the time of ruin and the deficit at ruin by solving a class of Seal's type partial integro-differential equations. Dickson [6] shows that the solution to the Seal's type equation obtained in [37] can be used to find simpler and interpretable expressions for some t-deferred ruin related probabilities (probability of ruin related events with ruin occurring after time t). Inspired by their work, in this paper we intend to show that the ideas and approaches proposed in their papers can be used to find expressions for some more complicated distribution functions and joint distribution functions; they are the joint distribution of the time of ruin and the maximum surplus prior to ruin, the joint distribution of the time of ruin and the maximum severity of ruin, and the distribution of the two-sided first exit time. Moreover, these approaches can also be applied to study and derive an explicit expression for the finite-time expected discounted penalty function (also called finite-time Gerber–Shiu function) in the sense that ruin occurs no later than finite time t. See Section 6 for more details on different definitions of the finite-time Gerber–Shiu function and relevant references. Note that the explicit expressions here, as in [37] and [6], are all in terms of known functions and integrations of these known functions such as the ultimate ruin probability and the ordinary (infinite time) Gerber–Shiu function introduced in [12].

The main steps and approaches used in this paper can be described as follows. Instead of considering the finite-time ruin related quantities directly, we consider the t-deferred ones (with the time of ruin occurring after time t) including joint distribution functions and finite-time Gerber–Shiu functions, as in [6]. We then prove that these t-deferred quantities satisfy their corresponding Seal's type partial integro-differential equation in two variables (initial surplus level u and time t). By applying the main results (see Section 2) from [37], we are able to express these t-deferred functions in terms of known functions and then the finite-time ruin related quantities that we look for can be obtained through their relationships with corresponding t-deferred ones. Because the solution to the Seal's type equation also depends on its initial value when $u = 0$,