Applied Mathematics and Computation 315 (2017) 400–413

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Rank/inertia approaches to weighted least-squares solutions of linear matrix equations

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ARTICLE INFO

MSC:
15A09
15A24
62J05
62H12

Keywords:
Matrix equation
Weighted least-squares solution
Quadratic matrix-valued function
Rank
Inertia
Optimization

ABSTRACT

The well-known linear matrix equation $AX = B$ is the simplest representative of all linear matrix equations. In this paper, we study quadratic properties of weighted least-squares solutions of this matrix equation. We first establish two groups of closed-form formulas for calculating the global maximum and minimum ranks and inertias of matrices in the two quadratical matrix-valued functions $Q_1 - XP_1X'$ and $Q_2 - X'P_2X$ subject to the restriction $\operatorname{trace}[(AX - B)'W(AX - B)] = \min$, where both P_i and Q_i are real symmetric matrices, $i = 1, 2$, W is a positive semi-definite matrix, and X' is the transpose of X. We then use the rank and inertia formulas to characterize quadratic properties of weighted least-squares solutions of $AX = B$, including necessary and sufficient conditions for weighted least-squares solutions of $AX = B$ to satisfy the quadratic symmetric matrix equalities $XP_1X' = Q_1$ an $X'P_2X = Q_2$, respectively, and necessary and sufficient conditions for the quadratic matrix inequalities $XP_1X' \succ Q_1$ ($\succcurlyeq Q_1, \prec Q_1, \preccurlyeq Q_1$) and $X'P_2X \succ Q_2$ ($\succcurlyeq Q_2, \prec Q_2, \preccurlyeq Q_2$) in the Löwner partial ordering to hold, respectively. In addition, we give closed-form solutions to four Löwner partial ordering optimization problems on $Q_1 - XP_1X'$ and $Q_2 - X'P_2X$ subject to weighted least-squares solutions of $AX = B$.

1. Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the set of all $m \times n$ real matrices. A', $r(A)$, and $\mathscr{R}(A)$ stand for the transpose, rank, and range (column space) of a matrix $A \in \mathbb{R}^{m\times n}$, respectively. I_m denotes the identity matrix of order m. $[A, B]$ denotes a row block matrix consisting of A and B. The Moore–Penrose inverse of $A \in \mathbb{R}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution X satisfying the four matrix equations $AXA = A$, $XAX = X$, $(AX)' = AX$, and $(XA)' = XA$. E_A and F_A stand for $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$ with $r(E_A) = m - r(A)$ and $r(F_A) = n - r(A)$. The Frobenius norm of a matrix $A \in \mathbb{R}^{m\times n}$ is defined to be $\|A\| = \sqrt{\operatorname{trace}(AA')}$. The symbols $i_+(A)$ and $i_-(A)$ for $A = A' \in \mathbb{R}^{m\times m}$, called the partial inertia of A, denote the number of the positive and negative eigenvalues of A counted with multiplicities, respectively. For brief, we use $i_\pm(A)$ to denote the both numbers. For a symmetric matrix $A = A' \in \mathbb{R}^{m\times m}$, the notations $A \succ 0$, $A \succcurlyeq 0$, $A \prec 0$, and $A \preccurlyeq 0$ mean that A is positive definite, positive semi-definite, negative definite, and negative semi-definite, respectively. Two symmetric matrices A and B of the same size are said to satisfy the inequalities $A \succ B$, $A \succcurlyeq B$, $A \prec B$, and $A \preccurlyeq B$ in the Löwner partial ordering if $A - B$ is positive definite, positive semi-definite, negative definite, and negative semi-definite respectively. It is well known that

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http://dx.doi.org/10.1016/j.amc.2017.07.079
0096-3003/

the Löwner partial ordering is a surprisingly strong and useful relation between two complex Hermitian (real symmetric) matrices.

We next present an motivation in statistics for our approach on the applications of the matrix rank/inertia methodology to weighted least-squares solutions (WLSSs) of linear matrix equations. Consider a general linear model defined by

$$y = X\beta + e, \quad \mathrm{E}(e) = 0, \quad \mathrm{Cov}(e) = \sigma^2\Sigma, \tag{1.1}$$

where y is an $n \times 1$ observable random vector, X is an $n \times p$ known matrix of arbitrary rank, β is a $p \times 1$ fixed but unknown parameter vector, e is a random error vector, σ^2 is an arbitrary positive scaling factor, and Σ is an $n \times n$ known positive semi-definite matrix of arbitrary rank. Recall that the method of least squares is often used to generate estimators and other statistics in regression analysis. Under the assumption in (1.1), the well-known ordinary least-squares estimator (OLSE) of the unknown parameter vector β in (1.1) is defined to be

$$\widehat{\beta} = \underset{\beta \in \mathbb{R}^{p\times 1}}{\mathrm{argmin}}(\, y - X\beta\,)'(\, y - X\beta\,), \tag{1.2}$$

while the OLSE of the vector $K\beta$ of parametric functions under (1.1) is defined to be $K\widehat{\beta}$. An alternative definition of the OLSE of β in (1.1) is given by

$$\widehat{\beta} = \widehat{L}y, \quad \text{where} \;\; \widehat{L} = \underset{L \in \mathbb{R}^{p\times n}}{\mathrm{argmin}}(\, y - XLy\,)'(\, y - XLy\,). \tag{1.3}$$

Furthermore, assume that V is a known positive semi-definite matrix. The weighted least-squares estimator (WLSE) of the unknown parameter vector β in (1.1) is defined to be

$$\widehat{\beta} = \underset{\beta \in \mathbb{R}^{p\times 1}}{\mathrm{argmin}}(\, y - X\beta\,)'V(\, y - X\beta\,), \tag{1.4}$$

while the WLSE of the parametric vector $K\beta$ under (1.1) is defined to be $K\widehat{\beta}$. The method of least squares in statistics is a standard approach for estimating unknown parameters in linear statistical models, which was first proposed as an algebraic procedure for solving overdetermined systems of equations by Gauss (in unpublished work) in 1795 and independently by Legendre in 1805, as remarked in [1,8,21]. It is well known that OLSEs/WLSEs based on the above norm minimization problems play an essential role in statistical inference of linear models and have been important objects of study in estimation theory of regression analysis.

Assume now that a linear matrix equation is given by

$$AX = B, \tag{1.5}$$

where $A \in \mathbb{R}^{p\times n}$ and $B \in \mathbb{R}^{p\times m}$ are two given matrices, and $X \in \mathbb{R}^{n\times m}$ is unknown. Similar to (1.4), the weighted least-squares solution (WLSS) of (1.5) is defined to a solution X of the following matrix trace minimization problem

$$\|\, AX - B\,\|_W^2 = \mathrm{tr}[\,(AX - B\,)'W(\,AX - B\,)\,] = \min, \tag{1.6}$$

where $W \in \mathbb{R}^{p\times p}$ is a positive semi-definite and is called the weighted matrix. Also denote

$$\mathcal{S} = \big\{X \in \mathbb{R}^{n\times m} \mid \mathrm{tr}[\,(AX - B\,)'W(\,AX - B\,)\,] = \min\big\}. \tag{1.7}$$

In order to approach quadratic algebraic properties of WLSSs of (1.5), we construct two quadratic matrix-valued functions from WLSSs of (1.5) as follows

$$\phi_1(X) = Q_1 - XP_1X', \qquad \phi_2(X) = Q_2 - X'P_2X, \tag{1.8}$$

where $P_1 = P_1' \in \mathbb{R}^{m\times m}$, $Q_1 = Q_1' \in \mathbb{R}^{n\times n}$, $P_2 = P_2' \in \mathbb{R}^{n\times n}$, and $Q_2 = Q_2' \in \mathbb{R}^{m\times m}$ are given matrices. Under the assumptions in (1.5)–(1.8), we propose the following several problems on matrix rank/inertia optimization, as well as matrix equalities/inequalities in the Löwner partial ordering.

Problem 1.1. Under the assumptions in (1.5)–(1.8), establish closed-form formulas for calculating the following max-min ranks/inertias

$$\max r(\,Q_1 - XP_1X'\,) \quad \text{s.t.} \quad \|\, AX - B\,\|_W = \min, \tag{1.9}$$

$$\min r(\,Q_1 - XP_1X'\,) \quad \text{s.t.} \quad \|\, AX - B\,\|_W = \min, \tag{1.10}$$

$$\max i_{\pm}(\,Q_1 - XP_1X'\,) \quad \text{s.t.} \quad \|\, AX - B\,\|_W = \min, \tag{1.11}$$

$$\min i_{\pm}(\,Q_1 - XP_1X'\,) \quad \text{s.t.} \quad \|\, AX - B\,\|_W = \min, \tag{1.12}$$