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A meshless discrete collocation method for the numerical solution of singular-logarithmic boundary integral equations utilizing radial basis functions

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ABSTRACT

The main intention of the current paper is to describe a scheme for the numerical solution of boundary integral equations of the second kind with logarithmic singular kernels. These types of integral equations result from boundary value problems of Laplace's equations with linear Robin boundary conditions. The method approximates the solution using the radial basis function (RBF) expansion with polynomial precision in the discrete collocation method. The collocation method for solving logarithmic boundary integral equations encounters more difficulties for computing the singular integrals which cannot be approximated by the classical quadrature formulae. To overcome this problem, we utilize the non-uniform composite Gauss–Legendre integration rule and employ it to estimate the singular logarithm integrals appeared in the method. Since the scheme is based on the use of scattered points spread on the analyzed domain and does not need any domain elements, we can call it as the meshless discrete collocation method. The new algorithm is successful and easy to solve various types of boundary integral equations with singular kernels. We also provide the error estimate of the proposed method. The efficiency and accuracy of the new approach are illustrated by some numerical examples.

1. Introduction

Boundary value problems for two-dimensional Laplace's equations with the linear Robin boundary conditions, i.e.

$$\begin{cases} \Delta u(\mathrm{x}) = 0, & \mathrm{x} \in D \subset \mathbb{R}^2, \\ \dfrac{\partial u(\mathrm{x})}{\partial n_{\mathrm{x}}} + p(\mathrm{x})u(\mathrm{x}) = g(\mathrm{x}), & \mathrm{x} \in \partial D, \end{cases} \tag{1}$$

using Green's formula can be reduced to second-kind boundary integral equations with logarithmic kernels [11,23], given in the form

$$-\pi u(\mathrm{x}) + \int_{\partial D} u(\mathrm{y}) \left(p(\mathrm{y}) \ln|\mathrm{x}-\mathrm{y}| + \frac{\partial \ln|\mathrm{x}-\mathrm{y}|}{\partial n_{\mathrm{y}}} \right) ds_{\mathrm{y}} = \int_{\partial D} g(\mathrm{y}) \ln|\mathrm{x}-\mathrm{y}| ds_{\mathrm{y}}, \qquad \mathrm{x} \in \partial D, \tag{2}$$

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where the region D is open, bounded and simply connected in $\mathbb{R}^2$, n_x denotes the outward unit normal vector on ∂D, $p(x)$ and $g(x)$ are given functions on ∂D with $p(x) \geq 0$ but $p \not\equiv 0$ and the unknown function $u(x)$ must be determined [11,23]. It should be noted that these types of integral equations are also appeared in connection with other types of partial differential equations (PDEs) occurred in several branches of sciences such as electromagnetism, solid and fluid mechanics, seismology, atomic scattering, wave diffraction, heat transfer, etc [11–13,19,43,48].

Solving analytically boundary integral equations, especially in the singular case, is mostly difficult so, it is significant to obtain their numerical solutions. The Galerkin and collocation methods [9,11] are the high usage schemes for the approximate solutions of boundary integral equations which use a family of functions as basis. Authors of [10] have investigated the piecewise polynomial collocation method for solving boundary integral equations. Wavelets, as very well localized functions [1,15,29], have been used to obtain the numerical solutions of boundary integral equations with logarithmic singular kernels in [23,27,45,51]. An iterative quadrature method [50] has been presented for the boundary integral equation deduced as reformulation of some boundary value problems for the two-dimensional Helmholtz equation. A spectral Galerkin method [41] has been applied to solve the boundary integral equation which arises from the two-dimensional Dirichlet problem. Fourier–Nystrom discretization scheme [31] has been developed for boundary integral equations associated with the Helmholtz equation. A meshless discrete Galerkin method [6] has been utilized to solve logarithmic boundary integral equations based on the moving least squares (MLS) approximation.

Meshless methods are based upon the scattered data approximations [17,18] that estimate a function without mesh generation on the domain. These methods have gained more attention not only by mathematicians but also in the engineering community. Among meshless methods, RBFs have become known as a powerful tool for the scattered data interpolation problems. The main advantage of RBFs is that they involve a single independent variable regardless of the dimension of the problem [49]. In recent years, the main focus of the applications of meshless methods especially based on the use of the RBFs seems to have shifted from scattered data interpolation to the numerical solution of PDEs. One of the domain-type meshless methods, the so-called Kansa's method developed by Kansa in 1990 [34,35], is obtained by directly collocating the RBFs, particularly the multiquadric, for the numerical approximation of the solution. Kansa's method was recently extended to solve various ordinary and partial differential equations [20,32,40,46,47].

We would like to review some of the most recent works for the numerical solution of integral equations via the meshless approximations. The meshless discrete collocation schemes have been investigated based on the RBFs for solving linear and nonlinear integral equations on non-rectangular domains with sufficiently smooth kernels [2,3] and weakly singular kernels [7,8]. The RBFs have been applied for the numerical solution of one-dimensional linear Fredholm and Volterra integral equations [28] and nonlinear Volterra–Fredholm–Hammerstein integral equations [44]. The meshless product integration (MPI) method [5] has been proposed to solve one-dimensional linear weakly singular integral equations. The MLS methodology as a local meshless method has been used for solving linear and nonlinear two-dimensional integral equations on non-rectangular domains [4,42] and integro-differential equations [21]. The paper [6] has described a computational method for solving Fredholm integral equations with logarithmic kernels so-called the meshless discrete Galerkin (MDG) method. Authors of [37,39] have introduced a MLS-based meshless Galerkin method for boundary integral equations and provided the error bound for the method [38].

The main purpose of this article is to propose a method for obtaining the numerical solution of the logarithmic singular boundary integral equation of the second kind (2). The method utilizes the extension of RBFs constructed on distributed nodal points to approximate the unknown function in the discrete collocation method. The discrete collocation method for solving singular integral equations usually needs a special integration rule to approximate its integrals. We will establish this quadrature scheme utilizing the composite non-uniform Gauss–Legendre quadrature formula. The scheme reduces the solution of the boundary Fredholm integral equation to the solution of a system of linear algebraic equations. The proposed scheme is a meshless method, since it requires no domain elements for interpolation or approximation. We also obtain the error bound and the rate of convergence for the proposed method. The new technique is simple, efficient, computationally attractive and more flexible for most classes of boundary integral equations.

The outline of the current paper is as follows: In Section 2, we review some basic formulations and properties of the RBFs method. In Section 3, we present a computational method for solving the boundary integral equation (2) by combining the RBF interpolation and the collocation method. In Section 4, we provide the error analysis for the proposed method. Numerical examples are given in Section 5. Finally, we conclude the article in Section 6.

2. Radial basis functions

RBFs were first developed by Hardy [30] in 1971 as a multidimensional scattered interpolation method in modeling of the earth's gravitational field. It was not recognized by most academic researchers until Franke [26] published a review paper in the evaluation of two-dimensional interpolation methods. We introduce RBFs for interpolating functions on $D \subset \mathbb{R}^d$ for every d. Before that, radial functions are defined as follows [25,49]:

Definition 2.1. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is called to be radial if there exists a univariate function $\phi : [0, \infty) \to \mathbb{R}$ such that

$$\Phi(\mathbf{x}) = \phi(r), \tag{3}$$

where $r = \|\mathbf{x}\|$ and $\|.\|$ is some norm on $\mathbb{R}^d$, usually the Euclidean norm.