Applied Mathematics and Computation 315 (2017) 468–476

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On exponential of split quaternionic matrices

Melek Erdoğdu[a,*], Mustafa Özdemir[b]

[a] Department of Mathematics, Konya Necmettin Erbakan University, Konya 42060, Turkey
[b] Department of Mathematics, Akdeniz University, Antalya 07058, Turkey

ARTICLE INFO

Keywords:
Quaternion
Split quaternion
Quaternion matrix

ABSTRACT

The exponential of a matrix plays an important role in the theory of Lie groups. The main purpose of this paper is to examine matrix groups over the split quaternions and the exponential map from their Lie algebras into the groups. Since the set of split quaternions is a noncommutative algebra, the way of computing the exponential of a matrix over the split quaternions is more difficult than calculating the exponential of a matrix over the real or complex numbers. Therefore, we give a method of finding exponential of a split quaternion matrix by its complex adjoint matrix.

1. Introduction

The matrix exponential is a matrix function on square matrices similar to the ordinary exponential function. Not only, the matrix exponential enters into the definition of the Lie algebra of a matrix Lie group but also it is the mechanism for passing information from the Lie algebra to the Lie group. Briefly, the matrix exponential gives a kind of connection between any matrix Lie algebra and the corresponding Lie group. Since many computations can be done much more easily at the level of the Lie algebra, the matrix exponential is of the first order of importance in studying (matrix) Lie groups.

Let A be an $n\times n$ real or complex matrix. The matrix exponential of A, denoted by e^A, is the $n\times n$ matrix given by the power series

$$e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^n.$$

The above series always converges, so the exponential of A is well-defined. In principle, the exponential of a matrix could be computed in many ways. Methods involving approximation theory, differential equations, the matrix eigenvalues, and the matrix characteristic polynomial have been proposed. Note that when $n=1$, the definition of the matrix exponential converts to the usual definition of the exponential for numbers. The question is whether the matrix exponential satisfies the same properties as the number exponential. The answer depends on the commutativity of matrices. For example, the argument $e^{a+b}=e^a e^b$ is an identity for any real or complex numbers a and b. But at this point we have to use the commutativity of multiplication to rewrite the second term. We cannot do this in general for matrices. If the matrices A and B commute, so that $AB=BA$, then we can easily see that $e^{A+B}=e^A e^B$. Another key point is the case of matrix exponential for the noncommutative algebras. Since the set of quaternions, which is first described by Irish Mathematician Sir William Rowan Hamilton as

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\}$$

* Corresponding author.
E-mail addresses: merdogdu@konya.edu.tr, melekrdgd@gmail.com (M. Erdoğdu), mozdemir@akdeniz.edu.tr (M. Özdemir).

http://dx.doi.org/10.1016/j.amc.2017.08.007
0096-3003/

with the relations $i^2 = j^2 = k^2 = -1$ and $ijk = -1$, is the most important member of noncommutative algebras, then there are a lot of works associated with quaternion matrices. First of all, a brief survey on quaternions and matrices of quaternions is presented in the study [15]. As a result of noncommutativity of quaternions, the theory eigenvalues of a quaternion matrix becomes one of the interesting topics about matrices over quaternions. Not only, the right eigenvalues for quaternionic matrices with a topological approach has been discussed by Baker in [4], but also, a study on left eigenvalues of a quaternionic matrix has been done by Huang and So [10]. On the other hand, some differences between right and left eigenvalues of quaternion matrices has been given in [16], including the Gershgorin type theorems for right and left eigenvalues of quaternionic matrices. Also, some differences of eigenvalue problem of complex and quaternion matrices have been discussed in [9] by Farid et al. Besides, an isomorphism between matrix algebras and simple orthogonal Clifford algebras is used to compute exponential of matrices over real, complex numbers and quaternions in the study [1]. Furthermore, an interesting method is constructed to compute exponential of quaternionic matrix by the transformation Ψ_n from quaternionic square matrices to complex square matrices in the study [12].

Soon after Hamilton's discovery of quaternions, James Cockle introduced the set of split quaternions as

$$\widehat{\mathbb{H}} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\}$$

with the relations $i^2 = -1,\ j^2 = k^2 = 1$ and $ijk = 1$. The set of split quaternions is noncommutative, too. Contrary to quaternion algebra, the set of split quaternions contains zero divisors, nilpotent elements and nontrivial idempotents [11,13,14]. Since, the rotations in Minkowski 3 space can be stated with split quaternions such as expressing the Euclidean rotations using quaternions, then there are works on geometric applications of split quaternions such as [13] and [11]. On the other hand, split quaternionic matrices and the complex adjoint matrix are introduced in [2]. Then, eigenvalues of a split quaternion matrix are discussed in [6] by using its complex adjoint matrix. Besides, the complex split quaternions and their matrices are investigated in [7]. Moreover, a Lorentzian transformation is represented as 2×2 split quaternion unitary matrix in [3]. Finally, the matrices over dual split quaternions investigated by their split quaternion matrix representation in [8].

The present article is concerned with computing exponential of split quaternionic matrices. First, a brief summary of split quaternion and some essential properties of split quaternionic matrices are given to necessary background. Then, we give a way of finding exponential of a split quaternion matrix by using its complex adjoint matrix. Moreover, we examine the case of single split quaternion in two different ways. First way is obtained by the relation between the exponential of the split quaternion and its complex adjoint matrix. The second way is a result of the concept of split quaternion. Also, we give examples of these two ways. Finally, we present some results about matrix exponential of a split quaternionic matrix.

2. Preliminaries

The set of split quaternions can be represented as

$$\widehat{\mathbb{H}} = \{q = q_0 + q_1 i + q_2 j + q_3 k : q_0,\ q_1,\ q_2,\ q_3 \in \mathbb{R}\}$$

where the imaginary units satisfy the relations

$$\begin{aligned} &i^2 = -1,\ \ j^2 = k^2 = 1, \\ &ij = -ji = k,\ \ jk = -kj = -i,\ \ ki = -ik = j. \end{aligned}$$

For any $q = q_0 + q_1 i + q_2 j + q_3 k \in \widehat{\mathbb{H}}$, we define scalar and vector part of q as $S(q) = q_0$ and $V(q) = q_1 i + q_2 j + q_3 k$, respectively. The conjugate of a split quaternion q is denoted by $\overline{q}$ and it is

$$\overline{q} = S(q) - V(q) = q_0 - q_1 i - q_2 j - q_3 k.$$

And the norm of the split quaternion q is defined by

$$\|q\| = \sqrt{|q\overline{q}|} = \sqrt{\left|q_0^2 + q_1^2 - q_2^2 - q_3^2\right|}.$$

The sum and product of split quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ are

$$\begin{aligned} p + q &= S(p) + S(q) + V(p) + V(q), \\ pq &= S(p)S(q) + S(p)V(q) + S(q)V(p) + \langle V(p), V(q)\rangle_{\mathbb{L}} + V(p) \wedge_{\mathbb{L}} V(q), \end{aligned}$$

respectively. Here $\langle\ ,\ \rangle_{\mathbb{L}}$ and $\wedge_{\mathbb{L}}$ denote Lorentzian inner and vector product and are defined as

$$\langle u, v\rangle_{\mathbb{L}} = -u_1 v_1 + u_2 v_2 + u_3 v_3,$$

$$u \wedge_{\mathbb{L}} v = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ of Minkowski 3 space, respectively. For any $q \in \widehat{\mathbb{H}}$ there exists a unique representation of the form $q = c_1 + c_2 j$ such that c_1, $c_2 \in C$. For more information about split quaternions, the readers are referred to the papers [11,13,14].