Applied Mathematics and Computation 315 (2017) 564–590

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the dynamics of a triparametric family of optimal fourth-order multiple-zero finders with a weight function of the principal m^{th} root of a function-to function ratio

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ARTICLE INFO

MSC:
65H05
65H99
41A25
65B99

Keywords:
Parameter space
Möbius map
Dynamical plane
Multiple-zero
Fourth-order
Conjugacy

ABSTRACT

Under the assumption of known root multiplicity $m \in \mathbb{N}$, a triparametric family of two-point optimal quartic-order methods locating multiple zeros are investigated in this paper by introducing a weight function dependent on a function-to-function ratio. Special cases of weight functions with selected free parameters are considered and studied through various test equations and numerical experiments to support the theory developed in this paper. In addition, we explore the relevant dynamics of proposed methods via Möbius conjugacy map when applied to a prototype polynomial $(z-a)^m(z-b)^m$. The results of such dynamics are visually illustrated through a variety of parameter spaces as well as dynamical planes.

1. Introduction

It is not surprising to acknowledge that one of the important tasks is accurately solving nonlinear governing equations of the form $f(x) = 0$ occurring in many fields of sciences and engineering technologies. Since exact methods are rarely available in most of real-life application problems such as weather forecast, satellite orbit data calculations and the control of global positioning systems, we usually resort to some stable numerical methods, one of which is widely known as Newton's method for simple-zero finders. Given the root multiplicity m, keeping in mind that the importance of theoretical study on multiple-zero finders has been emphasized by Traub [41], we will develop a family of optimal quartic-order multiple-zero finders to be derived in Section 3 by extending the well known modified Newton's method below:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

for a good initial guess x_0 chosen near the zero α. It is known that numerical scheme (1.1) is a second-order one-point optimal [28] method on the basis of Kung-Traub's conjecture [28] that any multipoint method [24,25,39] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. Other higher-order methods for nonlinear equations will be referred in Section 2.

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http://dx.doi.org/10.1016/j.amc.2017.08.005

As can be seen in (1.1), any numerical scheme can be symbolically written as

$$x_{n+1} = R_f(x_n), \ \ n = 0, 1, 2, \dots, \tag{1.2}$$

where R_f is an iteration function or fixed point operator [14]. It is clear that the process of finding a numerical solution of $f(x) = 0$ can be represented as a sequence of images of initial guess x_0 under the action of R_f below:

$$\{x_0, R_f(x_0), R_f^2(x_0), \dots, R_f^n(x_0), \dots, \},$$

which can be treated as a discrete dynamical system. Another objective of this paper is to extensively study of the dynamics behind the proposed family of methods which will be discussed in Section 5.

The remaining part of this paper is composed of six sections. Following this introductory section, Section 2 shortly describes studies on existing multiple-zero finders. Investigated in Section 3 is methodology and convergence analysis for newly proposed zero finders. A main theorem is established to state convergence order of four as well as to derive asymptotic error constants and error equations by use of a family of weight functions dependent on a function-to-function ratio. In Section 4 special cases are considered with weight functions of polynomial and rational types of functions for some selected free parameters. Section 5 discusses the dynamics behind the fixed points of the proposed iterative maps. Extensively investigated are dynamical properties of the proposed methods along with illustrative description on stability analysis of their fixed points, parameter spaces and dynamical planes. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 8 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of typical existing methods. In the final section, we state the overall conclusion and briefly discuss possible future work developing higher-order methods by extending the current analysis.

2. Review of existing multiple-zero finders

Fourth-order root-finders for a given nonlinear equation $f(x) = 0$ have been developed by researchers such as Argyros et al. [8], Chun and Neta [16], Kou et al. [27], and many other researchers [20,21,25,26]. Special attention is paid to Jarratt-like [8] optimal fourth-order methods [19,29,40] respectively shown below by (2.1)–(2.3) and another family of optimal fourth-order methods (2.4) proposed by Liu [30]:

Kan:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{mf(x_n)}{2f'(x_n)}\dfrac{\{m^2(3m-2)f'(y_n)^2 - C_1 f'(x_n)f'(y_n) + C_2 f'(y_n)^2\}}{[(m-1)df'(x_n) - mf'(y_n)][mf'(y_n) - d(m+8)f'(x_n)]}, \ \ d = \left(\dfrac{m}{m+2}\right)^m, \\ \text{where } C_1 = md(6m^2 + 17m - 14) \text{ with } \ C_2 = d^2(3m^3 + 19m^2 + 16m + 16). \end{cases} \tag{2.1}$$

Sol:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{4md}{m^2 \upsilon - d(m^2 + 2m - 4)}\Big[1 - \dfrac{m^3(m-2)}{16d^2}(\upsilon - \dfrac{m+2}{m}d)^2\Big] \cdot \dfrac{f(x_n)}{f'(x_n)}, \ \ \upsilon = \dfrac{f'(y_n)}{f'(x_n)}, \ \ d = (\frac{m}{m+2})^m. \end{cases} \tag{2.2}$$

Li:

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{m}{2}[\dfrac{(m-2)\upsilon - md}{d - \upsilon}] \cdot \dfrac{f(x_n)}{f'(x_n)}, \ \ d = (\dfrac{m}{m+2})^m, \ \ \upsilon = \dfrac{f'(y_n)}{f'(x_n)}. \end{cases} \tag{2.3}$$

Liu:

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - mG_f(w) \cdot \dfrac{f(x_n)}{f'(x_n)}, w = \sqrt[m-1]{\dfrac{f'(y_n)}{f'(x_n)}}, m > 1, \\ \text{where } \ G_f \text{ is sufficiently differentiable at 0 with } G_f(0) = 0, G_f'(0) = 1, G_f''(0) = \dfrac{4m}{m-1}. \end{cases} \tag{2.4}$$

Observe that two methods **Sol** and **Li** are identical when $m = 2$. Convergence behavior of existing methods (2.1)–(2.4) for various test equations will be compared later in Section 6 with proposed methods to be investigated in the next section.

3. Methodology and convergence analysis

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a multiple zero α with a given integer multiplicity of $m \geq 1$ and be analytic [1,23] in a small neighborhood of α. Then, given an initial guess x_0 sufficiently close to α, we propose in this paper a family of new