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The extrapolation methods based on Simpson's rule for computing supersingular integral on interval

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ABSTRACT

The Simpson's rule for the computation of supersingular integrals in boundary element methods is discussed, and the asymptotic expansion of error function is obtained. A series to approach the singular point is constructed. The extrapolation algorithm is presented and the convergence rate is proved. Some numerical results are also reported to confirm the theoretical results and show the efficiency of the algorithms.

1. Introduction

Accurate calculation of boundary element methods (BEM) arising in boundary integral equations has been a subject of intensive research in recent years. The formulation of certain classes of boundary value problems in terms of supersingular integral equations:

$$I(f,s) := \fint_a^b \frac{f(t)}{(t-s)^3} dt = g(s) \quad s \in (a,b) \tag{1.1}$$

have drawn lots of interests. In the literature, different definitions of singular integrals are found which can be shown to be the same. We mention the following one

$$\fint_a^b \frac{f(t)}{(t-s)^3} dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{s-\varepsilon} \frac{f(t)}{(t-s)^3} dt + \int_{s+\varepsilon}^b \frac{f(t)}{(t-s)^3} dt - \frac{2f(s)}{\varepsilon} \right\}, \quad s \in (a,b) \tag{1.2}$$

where $\fint_a^b$ denotes a supersingualr integral and s is the singular point. Here the supersingular integral is one order higher singularity than hypersingular integral.

One of the major problems arising from boundary element methods is how to evaluate such supersingular integral efficiently [19]. Numerous work has been devoted in developing efficient quadrature formulas for hypersingular integral such as the Gaussian method [6–8], the Newton–Cotes method [11,15–18,20], the transformation method [3,5] and some other methods [2,4,7,9,13,24]. Because of the high-order singularity of the kernels, the rules for Hadamard finite-part integrals (including hypersingular and supersingular integrals) are less accurate than their counterparts for Riemann integrals. Newton–Cotes rules for evaluating hypersingular integrals were firstly suggested by Linz [11]. The superconvergence phenomenon of trapezoidal rule and Simpson's rule for hypersingular integrals was found in [15,17], which showed that one higher order superconvergence rate than that in general case can be achieved if the singular point is located at some a prior known point.

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Then, the superconvergence for arbitrary degree Newton–Cotes rules of hypersingular integrals were studied in [16] and the superconvergence rate was $O(h^{k+1})$. Recently, Newton–Cotes rules and the corresponding superconvergence for evaluating hypersingular integrals on a circle were discussed in [22,23].

As an efficient method to improve the accuracy of boundary element analysis, Newton–Cotes rule has been intensively researched. Integrals with kernels beyond hypersingular have not been extensively studied. Du [4] studied the composite Simpson's rule and showed the optimal global convergence rate is $O(h)$. Then, Wu and Sun [15] studied the superconvergence of trapezoidal rule and the $O(h^2)$ superconvergence rate was obtained when the singular point is located at the middle point of each subinterval away from two endpoints. Recently, Zhang et al. [21] discussed the superconvergence phenomenon of the composite Simpson's rule and also the $O(h^2)$ rate was obtained for those superconvergence points away from the endpoints.

Extrapolation methods as an accelerating convergence technique has been applied to many fields in computational of mathematics [12,14]. In the paper of Li et al. [10] , the trapezoidal rule for computation hypersingular integral by extrapolation methods was given. However, to our knowledge, no attempt has been made to apply extrapolation technique to accelerate convergence for the computation of supersingular integral.

As we know, the composite trapezoidal rule for the computing the supersingular integral does not converge in general, as the convergence rate is two order lower than the Riemann integrals. While the superconvergence rate for supersingular is $O(h^2)$ the same as the Riemann integrals, see Wu and Sun [15]. In this paper we focus on the asymptotic error expansion of the composite Simpson's rule for the computation of supersingular integrals. Based on the asymptotic error expansion

$$E_n(f) = \sum_{i=2}^{l-1} \frac{h^{i-1}}{2^{i-1}} f^{(i+1)}(s) a_i(\tau) + O(h^{l-2}), \tag{1.3}$$

where $a_i(\tau)$ are functions independent of h, and τ the local coordinate of the singular point. We suggest an extrapolation algorithm. For a given τ, a series of s_j is selected to approximate the singular point s accompanied by the refinement of the meshes. Moreover, by means of the extrapolation technique we not only obtain an approximation with higher order accuracy, but also get a posteriori error estimate.

The rest of this paper is organized as follows. In Section 2, after introducing some basic formulas of the general (composite) Simpson's rule, some notations and preliminaries, we present our main result. In Section 3, the proof is complete. In Section 4, extrapolation algorithm is presented. Finally, several numerical examples to validate our analysis.

2. Main result

Let $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a uniform partition of the interval $[a, b]$ with mesh size $h = (b - a)/n$. Define by $f_Q(t)$ the composite Simpson's rule for $f(t)$:

$$\begin{aligned} f_Q(t) = & \frac{2}{h^2}[f(t_{j-1})(t - t_{j-1/2})(t - t_j) \\ & - 2f(t_{j-1/2})(t - t_{j-1})(t - t_j) + f(t_j)(t - t_{j-1})(t - t_{j-1/2})], \end{aligned} \tag{2.1}$$

and a linear transformation

$$t = \hat{t}_j(\tau) := (\tau + 1)(t_j - t_{j-1})/2 + t_{j-1}, \quad \tau \in [-1, 1], \tag{2.2}$$

maps the reference element $[-1, 1]$ onto the subinterval $[t_{j-1}, t_j]$. Replacing $f(t)$ in (1.1) with $f_Q(t)$ gives the composite Simpson's rule:

$$I_n(f, s) := \int_a^b \frac{f_Q(t)}{(t - s)^3} dt = \sum_{j=0}^{2n} \omega_j(s) f(t_{j/2}) = I(f, s) - E_n(f), \tag{2.3}$$

where $\omega_j(s)$ denotes the Cotes coefficients, see [15], and $E_n(f)$ is the error functional.

We define

$$F_i(\tau) = \tau(\tau^2 - 1)[(\tau + 1)^i - 2\tau^2 + (\tau - 1)^i] \tag{2.4}$$

and

$$\phi_{i,i+2}(t) = \begin{cases} -\dfrac{1}{2} \displaystyle\int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle\int_{-1}^{1} \frac{F_i(\tau)}{\tau - t} d\tau, & |t| > 1. \end{cases} \tag{2.5}$$