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# The windowed scalogram difference: A novel wavelet tool for comparing time series

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ABSTRACT

We introduce a new wavelet-based tool called *windowed scalogram difference* (WSD), which has been designed to compare time series. This tool allows quantifying if two time series follow a similar pattern over time, comparing their scalograms and determining if they give the same weight to the different scales. The WSD can be seen as an alternative to another tool widely used in wavelet analysis called *wavelet squared coherence* (WSC) and, in some cases, it detects features that the WSC is not able to identify. As an application, the WSD is used to examine the dynamics of the integration of government bond markets in the euro area since the inception of the euro as a European single currency in January 1999.



## 1. Introduction

Quantifying relationships between time series has been historically one of the most frequently addressed issues by most scientific disciplines. A large number of mathematical and statistical methods have been developed and applied for measuring the strength and direction of relationships between time series. The great majority of these techniques have focused on the time domain. Correlation and regression analysis constitute the first and most popular tools to quantify the association between time series. Subsequently, a number of more sophisticated time series methods, including cointegration analysis [1], Granger causality tests [2], vector autoregressive (VAR) models [3] or generalized autoregressive conditional heteroscedasticity (GARCH) models [4,5] have been also used for the same purpose. In addition, several newly introduced techniques, such as the combined cointegration approach [6], the quantile-on-quantile method [7], the quantile correlation approach [8], the nonlinear autoregressive distributed lag (NARDL) model [9], or the quantile autoregressive distributed lag (QADL) method [10] are also very useful to assess the linkages among time series. An obvious limitation of these approaches is that they are restricted to one or at most two time scales, i.e., the short run and the long run. In some fields, such as economics and finance, traditional time domain models are insufficient to describe precisely the linkage between variables. For example, financial markets are complex systems consisting of thousands of heterogeneous agents making decisions over a different

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time frame (from minutes to years), so that the relationships between economic and financial variables may vary across time scales associated to different investment horizons of market participants (see [11]). To remedy this situation, a body of literature seeking to characterize the connection between time series at different frequencies has been also developed. The Fourier analysis represents the best exponent of this line of research focused on the frequency domain, although it has serious shortcomings. In particular, under the Fourier transform the time information is completely lost, so it is hard to distinguish transient relations or to identify structural changes. Therefore, this approach is not suitable for non-stationary processes (see [12]).

In this context, the wavelet theory is a very versatile methodology that allows to study a wide range of different signal properties. Due to this great flexibility, wavelet methods have been applied to many disciplines such as geophysics [13,14], meteorology [15,16], engineering [17,18], medicine [19,20], image analysis [21,22], economics [11,23], or, for instance, recently they have been used for measuring the degree of non-periodicity of a signal [24]. Hence, the wavelet analysis emerges as an appealing alternative to the Fourier transform that takes into account both time and frequency domains simultaneously, whose primary advantage is its ability to decompose any signal into time scale components. This property offers a unique opportunity to study relationships between time series in both, time and frequency domains, at the same time. In fact, wavelet techniques can reveal interactions which would be, otherwise, hard to detect by using any other statistical procedure.

The aim of this paper is to propose a novel wavelet-based tool, called *windowed scalogram difference* (WSD), which has been designed to compare time series. As its name suggests, this new measure is based on the concept of wavelet scalogram, restricted, however, to a finite window in time and scale. The main feature of the WSD is that it allows to assess whether two time series, measured preferably in the same units, follow a similar pattern over time and/or across scales (or frequencies) through the comparison of their respective scalograms for different windows in time and scale. The WSD can be regarded as an alternative tool to the widely applied *wavelet squared coherence* (WSC) [14,16], in the sense that both measures serve to evaluate the level of association between two time series, although from slightly different perspectives. As a matter of fact, in some cases (see Fig. 1), the WSD detects certain features that the WSC is not able to identify.

The paper is organized as follows. Section 2 introduces the concept of WSD, including some practical aspects and simulation results on the validity of this tool. In Section 3, the WSD is applied to real data to test its validity, examining the dynamics of the integration of government bond markets in the euro area since the inception of the euro in January 1999. Finally, Section 4 concludes the paper.

## 2. The windowed scalogram difference (WSD)

This section starts presenting some basic notions of wavelet theory and recalling the concept of wavelet scalogram. Subsequently, the concept of WSD is formally introduced as a tool for measuring the degree of similarity between two time series. Finally, some important practical aspects for the application of the WSD are discussed.

### 2.1. Basic concepts of Wavelets

A *wavelet* is a function $\psi \in L^2(\mathbb{R})$ with zero average (i.e., $\int_{\mathbb{R}} \psi = 0$), normalized ($\|\psi\| = 1$) and "centered" in the neighborhood of $t = 0$ [25]. Scaling $\psi$ by $s > 0$ and translating it by $u \in \mathbb{R}$, we can create a family of *time–frequency atoms* (also called *daughter wavelets*), $\psi_{u,s}$, as follows

$$\psi_{u,s}(t) := \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right). \tag{1}$$

Given a time series $f \in L^2(\mathbb{R})$, the *continuous wavelet transform* (CWT) of $f$ at time $u$ and scale $s$ with respect to the wavelet $\psi$ is defined as

$$Wf(u,s) := \langle f, \psi_{u,s}\rangle = \int_{-\infty}^{+\infty} f(t)\psi_{u,s}^*(t)\,\mathrm{d}t, \tag{2}$$

where $*$ denotes the complex conjugate. The CWT allows us to obtain the frequency components (or *details*) of $f$ corresponding to scale $s$ and time location $u$, thus providing a time–frequency decomposition of $f$.

On the other hand, the dyadic version of (1) is given by

$$\psi_{j,k}(t) := \frac{1}{\sqrt{2^k}}\psi\left(\frac{t-2^k j}{2^k}\right), \tag{3}$$

where $j, k \in \mathbb{Z}$ (note that there is an abuse of notation between (1) and (3), nevertheless the context makes it clear if we refer to (1) or (3)). It is important to construct wavelets so that the family of dyadic wavelets $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$. Thus, any function $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{j,k\in\mathbb{Z}} d_{j,k}\psi_{j,k}, \tag{4}$$