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Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain[☆]

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ABSTRACT

A Hadamard type fractional integro-differential equation on infinite intervals is considered. By using monotone iterative technique, we not only get the existence of positive solutions, but also seek the positive minimal and maximal solutions and get two explicit monotone iterative sequences which converge to the extremal solutions. At last, to illustrative the main result, an example is also discussed.

1. Introduction

In recent studies, fractional differential equation models have been paid more and more focus in theoretical and applied fields. It is really an important concern to a variety of fields including economics, signal, biophysics, electrical networks and so on. The tools of fractional differential equations is playing an significant role in mathematical analysis in describing real world materials. For the related applications and details about fractional differential equation, see some recent works in [1–19].

It's worth pointing out that it is Riemann–Liouville and Caputo type fractional derivatives that are given much priority in relative research. There is another kind of fractional derivative found in the literature due to Hadamard [20]. And Hadamard derivative involve logarithmic function of arbitrary exponent, which differs from the fractional derivatives of Caputo and Riemann–Liouville type. It's imperative to mention that the researches on Hadamard fractional differential equations are still at the early days and should be further studied. For detail about Hadamard fractional derivative and integral, see [21–25], some recent contributions can be found in [26–34].

By the use of some fixed point theorems, Ahmad and Ntouyas in [26] show some results about the existence and the uniqueness of solutions for a coupled system of Hadamard type fractional differential equations on bounded domain:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), & 1 < t < e, \quad 1 < \alpha \le 2 \\ D^{\beta}u(t) = g(t, u(t), v(t)), & 1 < t < e, \quad 1 < \beta \le 2 \\ u(1) = 0, & u(e) = I^{\gamma}u(\sigma_1), \\ v(1) = 0, & v(e) = I^{\gamma}v(\sigma_2), \end{cases} \tag{1.1}$$

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where $D^{(.)}$ denotes Hadamard fractional derivative of fractional order, I^{γ} denotes Hadamard fractional integral of order γ and $\gamma > 0, 0 < \sigma_1 < e, 0 < \sigma_2 < e$.

Recently in [27], Ahmad et al. considered a initial value problem of fractional integro-differential inclusions of Hadamard and Riemann–Liouville type on the limited interval:

$$\begin{cases} D^{\alpha}\left(x(t) - \sum\limits_{i=1}^{m} I^{\beta_i} h_i(t, x(t))\right) \in F(t, x(t)), \quad t \in J := [1, T], \\ u(1) = 0, \end{cases} \tag{1.2}$$

where D^{α} denotes the Hadamard fractional derivative of order $\alpha, 0 < \alpha \le 1, I^{\phi}$ is the Riemann–Liouville fractional integral of order $\phi > 0, \phi \in \{\beta_1, \beta_2, ..., \beta_m\}$.

In [28], Thiramanus et al. applied the idea of Leggett–Williams and Guo–Krasnoselkii's fixed point theorems to study the positive solution for Hadamard fractional differential equations on infinite intervals:

$$\begin{cases} D^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 1 < \alpha < 2, \quad t \in (1, +\infty), \\ u(1) = 0, \qquad D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta) \end{cases} \tag{1.3}$$

where D^{α} denotes the Hadamard fractional derivative of order $\alpha, \eta \in (1, +\infty)$ and I^{β_i} is the Hadamard fractional integral of order $\beta_i > 0,\ i = 1, 2, \ldots, m$ and $\lambda_i \ge 0,\ i = 1, 2, \ldots, m$ are given constants.

It's worth mentioning that Hadamard's construction is perfect for the problems containing half-line. As we all know, only the paper [28] studied the existence of solutions for Hadamard fractional differential equations on unbounded domain. On the other hand, on the existent literature [26–34], authors obtained only the existence results. Then a nature question arises "How to find it?" The thought always kept going round and round in authors' head. Motivated by the mentioned papers and the thought, the following boundary value problem of Hadamard fractional integro-differential equations on infinite domain is considered:

$$\begin{cases} {}^{H}D^{\alpha}u(t) + f(t, u(t), {}^{H}I^{r}u(t), {}^{H}D^{\alpha-1}u(t)) = 0, \quad 1 < \alpha < 2, \quad t \in (1, +\infty), \\ u(1) = 0, \qquad {}^{H}D^{\alpha-1}u(\infty) = \sum\limits_{i=1}^{m} \lambda_i {}^{H}I^{\beta_i}u(\eta) \end{cases} \tag{1.4}$$

where ${}^{H}D^{\alpha}$ denotes Hadamard fractional derivative of order α, $\eta \in (1, \infty)$ and ${}^{H}I^{(.)}$ is the Hadamard fractional integral, $r, \beta_i, \lambda_i \ge 0 (i = 1, 2, \ldots, m.)$ are given constants and α, η, β_i, λ_i satisfy $\Gamma(\alpha) > \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\alpha)}{\Gamma(\alpha+\beta_i)} (\log \eta)^{\alpha+\beta_i-1}$.

Compared with [28], in this paper, the function f contains Hadamard fractional integral operator ${}^{H}I^{r}u(t)$ and lower order derivative operator ${}^{H}D^{\alpha-1}u(t)$ as well. By the use of the monotone iterative method, not only could we get the existence of positive solutions for Hadamard fractional integro-differential equations on infinite intervals, but also we can seek the minimal and maximal positive solutions and get two explicit monotone iterative sequences converging to the extremal solution, which is different from the previous work [26–34]. It also successfully answer the above problem "How can we find a solution". For applications and details of the method, we refer to some works [35–51] and references cited therein.

The structure of this paper go on as follows: Section 2 contains some basic definitions and related lemmas that will be used. In Section 3, the main results and the proof are presented. In Section 4, main results are illustrated by an example.

2. Preliminaries

First of all, we recall some important definitions and related lemmas.

Now, we define two Banach spaces

$$E = \{u \in C([1, \infty)) : \sup_{t \in [1,\infty)} \frac{|u(t)|}{1 + (\log t)^{\alpha-1}} < \infty\},$$

$$F = \{u \in E : {}^{H}D^{\alpha-1}u(t) \in C([1, \infty)), \sup_{t \in [1,\infty)} |{}^{H}D^{\alpha-1}u(t)| < \infty\},$$

with norms $\|u\|_E = \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\alpha-1}}$ and $\|u\|_F = \max\{\|u\|_E, \sup_{t\in[1,\infty)} |{}^{H}D^{\alpha-1}u(t)|\}$, respectively.

Define a cone $P \subset F$ by

$$P = \{u \in F : u(t) \ge 0, t \in [1, \infty)\}.$$

Notate

$$\Omega = \Gamma(\alpha) - \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\alpha)}{\Gamma(\alpha + \beta_i)} (\log \eta)^{\alpha+\beta_i-1}, \tag{2.1}$$

obviously $\Omega > 0$.