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# SOR-like iteration method for solving absolute value equations☆

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ABSTRACT

In this paper, we propose an SOR-like iteration method for solving the absolute value equation (AVE), which is obtained by reformulating equivalently the AVE as a two-by-two block nonlinear equation. The convergence results of the proposed iteration method are proved under certain assumptions imposed on the involved parameter. Numerical experiments are given to demonstrate the feasibility, robustness and effectiveness of the SOR-like iteration method.



## 1. Introduction

Consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1.1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $|x|$ denotes the vector in $\mathbb{R}^n$ with absolute values of components of the vector $x$. The system (1.1) is a special case of the generalized absolute value equation of the following form

$$Ax + B|x| = b, \tag{1.2}$$

where $B \in \mathbb{R}^{n\times n}$, was introduced by Rohn [27] and further studied in [4,12,14–16,18,21,23–26,30].

The AVE (1.1) arises in many areas of scientific computing and engineering applications. For example, linear programs, quadratic programs and bimatrix games can be reduced to a linear complementarity problem (LCP) [7,25]. Based on the modulus method, LCP can be formulated as a system of absolute value equations such as (1.1) [21]. Specially, many modulus-based matrix splitting iteration methods are proposed for solving the solutions of LCP, see for example [2,5,13,31,34,35] and references therein.

In recent years, the problem of solving the AVE has attracted much attention and has been investigated in the literature, see for example [3,6,7,9–12,14–23,25–30,32,33,36] and references therein. A large variety of methods for solving the AVE in (1.1) have been developed. Most of those methods are based on the Newton algorithm as the AVE in (1.1) being a weakly nonlinear equation. For example, in [1], Mangasarian proposed a generalized Newton method for solving the AVE, which generates a sequence formally stated as

$$(A - D(x^{(k)}))x^{(k+1)} = b,$$

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where $D(x) := \mathrm{diag}(\mathrm{sgn}(x))$, $x \in \mathbb{R}^n$. In [25], Noor et al. proposed an iteration method for solving the AVE in (1.1) with $A$ being a symmetric matrix.

In this paper, by reformulating the AVE in (1.1) as a two-by-two block nonlinear equation, we propose an SOR-like iteration method for solving it, which is based on a splitting of the two-by-two block coefficient matrix. We prove that the proposed iteration method will converge to the solution of the AVE in (1.1) under suitable choices of the involved parameter. In addition, we also use numerical examples to show that the SOR-like iteration method is feasible and effective in computing.

This paper is organized as follows. In Section 2, we present some notations and preliminaries that will be used throughout the paper. In Section 3, we propose an SOR-like iteration method for solving the AVE in (1.1) and consider the convergence of the proposed iteration method. Experimental results and conclusions are given in Sections 4 and 5, respectively.

## 2. Preliminaries

In this section, we present some notations and auxiliary results.

Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ matrices with real entries and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. The $i$th component of a vector $x \in \mathbb{R}^n$ is denoted by $x_i$ for every $i = 1, \dots, n$. For $x \in \mathbb{R}^n$, $\mathrm{sgn}(x)$ denotes a vector with components equal to $-1$, 0 or 1 depending on whether the corresponding component of the vector $x$ is negative, zero or positive. Denote $|x|$ the vector with $i$th component equal to $|x_i|$.

The symbol $I$ denotes the $n \times n$ identity matrix. If $x \in \mathbb{R}^n$, then $\mathrm{diag}(x)$ will denote an $n \times n$ diagonal matrix with $(i, i)$th entry equal to $x_i$, $i = 1, \dots, n$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $m \times n$ matrices, $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ holds for all $1 \leq i \leq m$ and $1 \leq j \leq n$. A matrix $A \in \mathbb{R}^{m\times n}$ is said to be nonnegative (positive) if its entries satisfy $a_{ij} \geq 0$ $(a_{ij} > 0)$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. For the matrix $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the spectral norm defined by $\|A\| := \max\{\|Ax\| : x \in \mathbb{R}^n, \ \|x\| = 1\}$, where $\|x\|$ is the 2-norm.

The following proposition was proved in Proposition 4 of [21].

**Proposition 2.1** *[21]. Assume that $A \in \mathbb{R}^{n\times n}$ is invertible. If $\|A^{-1}\| < 1$, then the AVE in (1.1) has a unique solution for any $b \in \mathbb{R}^n$.*

About the nonnegative matrix, we have the following results.

**Lemma 2.1** *[3]. For any vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, the following results hold:*

(1) $\| |x| - |y| \| \leq \|x - y\|$;
(2) *if* $0 \leq x \leq y$, *then* $\|x\| \leq \|y\|$;
(3) *if* $x \leq y$ *and* $P$ *is a nonnegative matrix, then* $Px \leq Py$.

**Lemma 2.2** *[3]. For any matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, if $0 \leq A \leq B$, then $\|A\| \leq \|B\|$.*

## 3. SOR-like iteration method

Let $y = |x|$, then the AVE in (1.1) is equivalent to

$$\begin{cases} Ax - y = b, \\ -|x| + y = 0, \end{cases}$$

that is

$$\mathcal{A}\mathbf{z} := \begin{pmatrix} A & -I \\ -D(x) & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} := \mathbf{b}, \tag{3.1}$$

where $D(x) := \mathrm{diag}(\mathrm{sgn}(x))$, $x \in \mathbb{R}^n$.

Note that the Eq. (3.1) is also nonlinear, as the matrix $D(x)$ depends on the variable $x$. It is also quite complicated to solve (3.1) in actual computations.

Let

$$\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{U},$$

where

$$\mathcal{D} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} 0 & 0 \\ D(x) & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

then we can obtain the following fixed point equation

$$(\mathcal{D} - \omega\mathcal{L})\mathbf{z} = [(1-\omega)\mathcal{D} + \omega\mathcal{U}]\mathbf{z} + \omega\mathbf{b},$$

where the parameter $\omega > 0$. That is

$$\begin{pmatrix} A & 0 \\ -\omega D(x) & I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1-\omega)A & \omega I \\ 0 & (1-\omega)I \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \omega b \\ 0 \end{pmatrix}. \tag{3.2}$$