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# Linearized difference schemes for a BBM equation with a fractional nonlocal viscous term

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ABSTRACT

This paper is concerned with the efficient finite difference schemes for a Benjamin–Bona–Mahony equation with a fractional nonlocal viscous term. By using the weighted-shift Grünwald–Letnikov and the fractional centered difference formulae to approximate the nonlocal fractional operators, we design a class of linearized finite difference schemes for the presented nonlocal model. The existence, stability and convergence of the proposed numerical schemes are rigorously derived with the help of functional analysis. Theoretical analysis shows that the proposed numerical schemes are stable with second order accuracy. Numerical examples are presented to verify our theoretical analysis and to demonstrate the efficiency of the proposed numerical schemes.



## 1. Introduction

Benjamin–Bona–Mahony (BBM) equation describes the uni-directional propagation of surface water waves with small amplitudes and long wavelengths in nonlinear dispersive media [5]. It is well-known as a regularized counterpart of the Korteweg-de Vries equation and is extensively studied in the recent literature, see for example [5,6] and references therein. The nonlocal viscous term will arise in the BBM equation when the weak effects of dispersion and dissipation effects are considered for uni-directional wave propagation [6,20,25,29]. It usual appears in the following damped BBM equation [6,7]

$$u_t + u_x - \mu u_{txx} + \beta u^{\gamma} u_x = \mathcal{M}_x^{\alpha} u, \quad x \in (a,b), \quad t \in (0,T], \tag{1}$$

where $u = u(x,t)$ is the horizontal velocity of the fluid, $\gamma$ is a positive integer, $\mu$ and $\beta$ are non-negative parameters dedicated to the balance of viscosity and dispersion, $\mathcal{M}_x^{\alpha}$ is the nonlocal fractional operator

$$\mathcal{M}_x^{\alpha} u(x,t) = \kappa_1\, {}_aD_x^{\alpha} u(x,t) + \kappa_2\, {}_xD_b^{\alpha} u(x,t), \quad 1 < \alpha < 2, \tag{2}$$

with ${}_aD_x^{\alpha}$ and ${}_xD_b^{\alpha}$ are left and right Rieman–Liouville fractional derivatives of order $\alpha$, respectively, defined by

$${}_aD_x^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-\xi)^{1-\alpha} u(\xi,t) d\xi, \tag{3}$$

and

$${}_xD_b^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (\xi - x)^{1-\alpha} u(\xi,t) d\xi. \tag{4}$$

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The parameters $\kappa_1$ and $\kappa_2$ in Eq. (2) are non-negative constants which balance the effects of dispersion.

Many works have been proposed for the well-posedness and regularity of solutions for the nonlocal partial differential equations, see [1,2,13–15,19]. The effect of the viscous layer is modeled by a nonlocal term that acts as dissipation and dispersion, as revealed by the linear dispersion analysis [6,7,9]. These models are generalizations of the well-known BBM equation, in which the viscosity is not considered. In the past decade, many numerical methods for the linear space fractional partial differential equations have been extensively developed, see [8,12,18,21,23,24,27,28,30,32,34,38]. However, numerical methods for nonlinear fractional equations such as are far from being abundant. In the literature, based on the interacting particles approximation, Biler et al. [4] develop a numerical method for the solution of a large class of evolution problems involving the fractional Laplacian. Ervin et al. [17] developed a fully finite element approximation to solve a time dependent fractional nonlinear diffusion equation. Droniou [15] developed a class of finite difference schemes for a fractional Burgers equation. He proved that the numerical solutions converge towards to Alibaud's entropy solution. Cifani et al. [11] first developed a discontinuous Galerkin method for a fractional conservation law. Xu and Hesthaven [35] proposed a Runge–Kutta local discontinuous Galerkin method for a fractional conservation law, and they proved and stability and error estimations for the considered model. More recently, Chen et al. [10] discussed the time decay behavior of BBM equation with time and space nonlocal viscous terms by a Fourier spectral approximation [9,10,16]. Combining a linearized finite difference scheme for the time discretization and Fourier spectral method for the spatial discretization, Zhang and Xu in their recently work [39] constructed and analyzed two linearized schemes for the numerical solution of a time fractional water wave equation in BBM form.

Our interest in the present paper is to design efficient numerical schemes for the fractional BBM equation . Section 2 is devoted to discuss the linearized difference schemes for the fractional BBM equation. Section 3 contains numerical results for the considered equation, which demonstrate the accuracy and effectiveness of the proposed method. At last, we close by providing some concluding remarks in Section 4.

## 2. Second-order linearized difference schemes

We consider the difference schemes for Eq. in finite domain $(x, t) \in [a, b] \times [0, T]$. The computational domain $[a, b] \times [0, T]$ is divided into an $J \times N$ mesh with the spatial step size $h = (b-a)/J$ and the time step size $\tau = T/N$, respectively, where $J$ and $N$ are integers. Grid points $(x_j, t_n)$ are defined by $x_j = a + jh, 0 \le j \le J; t_n = n\tau, 0 \le n \le N$. Denote $u_j^n$ be the finite difference approximation of $u(x, t)$ at grid point $(x_j, t_n)$, i.e. $u_j^n \approx u(x_j, t_n)$ and $u_j^{n+\frac{1}{2}} = (u_j^n + u_j^{n+1})/2$. Denote the function space $W_h = \{(v_j), v_0 = 0, v_J = 0, 0 \le j \le J\}$. Suppose that $u^n = \{u_j^n | 0 \le j \le J\}$ and $v^n = \{v_j^n | 0 \le j \le J\}$ are two grid functions, we introduce the following notations

$$D_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad D_+ u_j^n = \frac{u_{j+1}^n - u_j^n}{h}, \quad D_- u_j^n = \frac{u_j^n - u_{j-1}^n}{h},$$

$$D_0 u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \quad D^2 u_j^n = D_+ D_- u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2},$$

$$(u^n, v^n)_h = \sum_{j=0}^{J-1} h u_j^n v_j^n, \quad ||u^n||_h = \sqrt{(u^n, u^n)}, \quad ||u^n||_\infty = \max_{0 \le j \le J-1} |u_j^n|, \tag{5}$$

$$|u^n|_{1,h}^2 = \sum_{j=0}^{J-1} h (D_+ u_j^n)^2, \quad ||u^n||_{1,h}^2 = ||u^n||_h^2 + |u^n|_{1,h}^2. \tag{6}$$

In the following, we will propose a second order finite difference scheme for the dBBM equation subject to the initial value

$$u(x, 0) = u_0(x), \quad x \in (a, b), \tag{7}$$

and boundary conditions

$$u(a, t) = 0, u(b, t) = 0, \quad t \in (0, T]. \tag{8}$$

In practical application, many different definitions of fractional derivatives are introduced [27,31]. The Grünwald–Letnikov derivative is more suitable to approximate the Riemann–Liouville derivative. However, the difference scheme based on the Grünwald–Letnikov formula for space fractional diffusion is unstable. To obtain a stable difference scheme, Meerschaert and Tadjeran [23] firstly introduced the shifted Grünwald–Letnikov approximation formula with only the first order accuracy. To improve the accuracy, Tian et al. [33] introduced the weighted and shifted Grünwald–Letnikov difference (WSGLD) operators for the Riemann–Liouville derivatives. For function $u(x) \in C^4[a, b]$ and all derivatives of $u$ up to order four belong to $L_1(a, b)$,