Applied Mathematics and Computation 311 (2017) 251–261

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Achieving higher order of convergence for solving systems of nonlinear equations ☆

Xiaoyong Xiao*, Hongwei Yin

Numerical Simulation and High-Performance Computing Laboratory, School of Sciences, Nanchang University, Nanchang 330031, PR China

ARTICLE INFO

MSC:
41A25
41A25
65H10
65Y04

Keywords:
Systems of nonlinear equations
Extended Newton iteration
Order of convergence
Higher order methods
Computational efficiency

ABSTRACT

In this paper, we develop a class of third order methods which is a generalization of the existing ones and a method of fourth order method, then introduce a technique that improves the order of convergence of any given iterative method for solving systems of nonlinear equations. Based on a given iterative method of order $p \geq 2$ which uses the extended Newton iteration as a predictor, a new method of order $p+2$ is proposed with only one additional evaluation of the function. Moreover, if the given iterative method of order $p \geq 3$ uses the Newton iteration as a predictor, then a new method of order $p+3$ can be developed. Applying this procedure, we obtain some new methods with higher order of convergence. Moreover, computational efficiency is analyzed and comparisons are made between these new methods and the ones from which have been derived. Finally, several numerical tests are performed to show the asymptotic behaviors which confirm the theoretical results.

1. Introduction

Solving systems of nonlinear equations is an important problem in numerical analysis and engineering. For a given nonlinear system $F(x)$: $D \subseteq R^n \to R^n$, we are going to find a vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^t$ such that $F(\alpha) = 0,$ where

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^t, \quad \text{and} \quad x = (x_{(1)}, x_{(2)}, \ldots, x_{(n)})^t.$$

As we all know, the most widely used algorithm to tackle this problem is the quadratically convergent Newton's method [19].

To accelerate the order of convergence of the existing iterative methods, a number of robust and efficient methods have been introduced in open literatures. For example, Cordero and Torregrosa [5,6], Frontini and Sormani [10], Grau-Sánchez et al. [11], Homeier [13,14], Noor and Waseem [17] have proposed third order methods each requiring the evaluations of two matrix inversions per iteration. Darvishi and Barati [8] have developed a third order method requiring the evaluations of two functions, one first derivative, and one matrix inversion. Cordero et al. [4] have introduced a fourth order method requiring the evaluations of two functions, two first derivatives, and one matrix inversion. Cordero et al. [3] have also implemented the fourth order Jarratt's method [15] for scalar equations to nonlinear systems, which requires one function, two first derivatives, and two matrix inversions. Darvishi and Barati [8] have proposed a fourth order method which uses two functions, three first derivatives, and two matrix inversions. Grau-Sánchez et al. [11] have introduced a fourth order method

☆ This work was supported by the National Natural Science Foundation of China (Grant Nos. 61563033 and 11401293).

* Corresponding author.
E-mail addresses: xiaoxy31@gmail.com (X. Xiao), hongwei-yin@hotmail.com (H. Yin).

http://dx.doi.org/10.1016/j.amc.2017.05.033
0096-3003/

which requires three functions, one first derivative, and one matrix inversion. Grau-Sánchez et al. [12] have also generalized the fourth Ostrowski's [18] method to systems of nonlinear equations requiring two functions, one first derivative, one divided difference and two matrix inversions per iteration. Sharma and Gupta [23] have extended the third Homeier's method [13] to be a fifth order method which uses two functions, two first derivatives, and two matrix inversions [1,7,21,22].

However, construction of higher order iterative methods should be based on low computational cost. Generally, the golden rule is to achieve as high as possible convergence order requiring as small as possible the evaluations of functions, derivatives and matrix inversions. Recently, some researchers focus on establishing a general law to accelerate the convergence for all the iterative methods which use the Newton iteration as a predictor. For the unidimensional case, Kou et al. [16] improved a third order method to be fifth order by the procedure

$$x_{k+1} = u_{k+1} - f(u_{k+1})/f'(y_k),$$

where $y_k = x_k - f(x_k)/f'(x_k)$, and $u_{k+1} = g_3(x_k)$ is the iteration function of a third order method. Cordero et al. [2] extended it to multidimensional case. They introduced a technique by which the order of convergence p of any given iterative method can be improved to $p+2$ by the construction

$$\begin{aligned} z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k), \end{aligned} \tag{1.1}$$

where $F'(x)^{-1}$ is the inverse of first Fréchet derivative $F'(x)$ of the function $F(x)$, $y_k = x_k - F'(x_k)^{-1}F(x_k)$ is the classic Newton iteration, and $z_k = \phi(x_k, y_k)$ is the iteration function of a method of order p, which uses $F'(y_k)$ in its iteration function. Applying this procedure, Cordero et al. [2] developed the following fifth order method extended from Frontini and Sormani [9]:

$$\begin{aligned} z_k &= x_k - 2[F'(y_k) + F'(x_k)]^{-1}F(x_k), \\ x_{k+1} &= z_k - F'(y_k)^{-1}F(z_k). \end{aligned} \tag{1.2}$$

Notice that the technique (1.1) is applicable only when Newton iteration is used as a predictor, however, there are a number of iterative constructions which use the extended Newton iteration as a predictor. For example, Homeier [13] proposed a third order method defined by

$$\begin{aligned} y_k &= x_k - \frac{1}{2}F'(x_k)^{-1}F(x_k), \\ z_k &= x_k - F'(y_k)^{-1}F(x_k). \end{aligned} \tag{1.3}$$

Sharma and Gupta [23] extended it to be a fifth order method by the construction

$$x_{k+1} = z_k - [2F'(y_k)^{-1} - F'(x_k)^{-1}]F(z_k). \tag{1.4}$$

where y_k and z_k are defined in (1.3). When the extended Newton iteration is used as a predictor, a technique to accelerate the convergence is introduced in [24]

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ z_k &= \phi(x_k, y_k), \\ x_{k+1} &= z_k - \left\{2\left[\frac{1}{2a}F'(y_k) + \left(1 - \frac{1}{2a}\right)F'(x_k)\right]^{-1} - F'(x_k)^{-1}\right\}F(z_k), \end{aligned} \tag{1.5}$$

by which the order of convergence of any given iterative method can be improved from p to $p+2$. Next, we are going to consider other techniques to accelerate the order of convergence.

The paper is organized as follows. In Section 2, we propose a class of third order methods which is a generalization of the existing ones and a method of fourth order method, and then a technique is introduced for improving the order of convergence of any given iterative method, which uses the extended Newton iteration as a predictor, from p to $p+2$ or $p+3$, with only one additional evaluation of the function. Some higher order methods are developed by applying this technique, computational efficiency is discussed and comparisons are made in Section 3. Several numerical examples are given in Section 4 to show the asymptotic behaviors of these methods. Finally, conclusions are made in Section 5.

2. Main result

Through careful observation of (1.2) and (1.3), we boldly introduce the following construction:

$$\begin{aligned} y_k &= x_k - aF'(x_k)^{-1}F(x_k), \\ x_{k+1} &= x_k - \left\{\left[bF'(y_k) + cF'(x_k)\right]^{-1} + dF'(x_k)^{-1}\right\}F(x_k), \end{aligned} \tag{2.1}$$

where the constants $a \neq 0$ and b, c, d are to be specified later. Thus the third order method by Frontini and Sormani [9] is the special case of $a = 1$, $b = c = 1/2$ and $d = 0$. The method by Homeier [13] given by (1.3) is the special case of $a = 1/2$, $b = 1$ and $c = d = 0$.