Applied Mathematics and Computation 311 (2017) 300–313

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A monotone finite volume scheme for diffusion equations on general non-conforming meshes

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ARTICLE INFO

Keywords:
Non-conforming
Vertex unknowns
Hanging nodes
Monotonicity
Diffusion equation

ABSTRACT

A nonlinear monotone finite volume scheme on general non-conforming meshes for diffusion equations is introduced, which deals with discontinuous tensor coefficients rigorously. Since the expression of normal flux depends on auxiliary unknowns defined at cell-vertex including hanging nodes, we propose a new method to eliminate vertex-unknown by using primary unknowns at the centers of the cells sharing the vertex. Especially the unknowns defined on hanging nodes are eliminated by flux continuous conditions. The resulting scheme is monotone and preserves positivity of analytical solutions for strongly anisotropic and heterogeneous full tensor coefficient problems. Numerical results show that the convergent order of the monotone scheme by different methods of eliminating vertex unknowns will vary remarkably, and our new method can assure that it has almost second order accuracy and more accurate than some existing methods.

1. Introduction

Consider the stationary diffusion problem for unknown $u = u(x)$:

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = f & \text{in } \Omega, \\ u(x) = g & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded polygonal set of R^2 with boundary $\partial\Omega$, and κ is diffusion tensor (possibly anisotropic and discontinuous).

For Lagrangian radiation hydrodynamic problems with high temperature and high pressure, large relative displacement is easy to appear near multi-material interface, which is named as sliding interface [23]. It follows that general non-conforming meshes (see Fig. 1) occur naturally since computational mesh moves with fluid flow. For multi-fluid Eulerian radiation hydrodynamic problems, adaptive mesh refinement (AMR) technique is popular [11,22], which results in the appearance of hanging nodes, and is used to increase resolution and reduce computational cost especially in the cases of physical quantities being with large gradients or discontinuities. Therefore, accurate and efficient discrete schemes on general non-conforming meshes are necessary for numerically solving these radiation hydrodynamic problems.

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http://dx.doi.org/10.1016/j.amc.2017.05.041

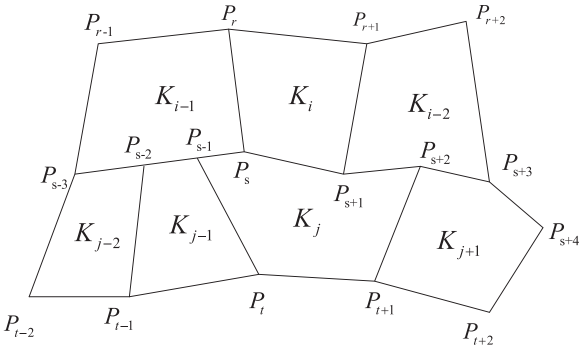

Fig. 1. Non-conforming mesh.

In [7] an improved cell-centered conservative scheme on orthogonal AMR meshes is presented to remove interface leading errors and ensure flux continuity. In [14] a mimetic finite difference (MFD) method on non-orthogonal locally refined quadrilateral meshes is proposed, and it is extended in [12] to polygonal, locally refined and non-matching meshes, and then a similar mimetic discretization on unstructured polyhedral meshes is given in [15]. A post-processing method for mimetic finite difference method of diffusion equations is presented in [3] on quite general meshes with non-convex and non-matching cells, and the post-processed solution is proved to be second order accurate under very general conditions. In [6] a mixed finite volume scheme for anisotropic diffusion equations on unstructured irregular (convex) meshes is constructed, which introduces three sets of unknowns, i.e., cell-centered unknowns u and discrete gradient ∇u in each cell, and discrete normal flux on each interior edge of cell. In [8] SHUSHI scheme on general meshes (including non-conforming grids) is actually derived from a discrete weak formulation, so it is a kind of non-conforming finite element method with cell-centered unknowns and some cell-edge unknowns to deal with diffusion tensor discontinuities. In [18] a second order cell-centered finite volume scheme for convection-diffusion equations on unstructured and non-conforming (convex) meshes is proposed, which uses SHUSHI scheme for diffusive fluxes with diffusion coefficients being a non-negative constant. In [4] a cell-centered finite volume scheme named as diamond scheme on general non-conforming meshes is presented, where a diamond stencil is used in deriving flux expression, and vertex unknowns are accurately approximated by a new weighted interpolation with weights being adaptive to both geometric parameters of cells and diffusion coefficients. In [9] a cell-centered scheme for the approximation of Laplace operator on non-conforming meshes is introduced for incompressible Navier–Stokes equations, which can be viewed as a low order non-conforming Galerkin approximation.

All schemes mentioned above are not monotone or cannot preserve positivity on general meshes. It is well known that classical finite volume (FV) and finite element (FE) schemes violate monotonicity for strong anisotropic diffusion tensors and on distorted meshes [5,10,17]. In order to assure the monotonicity, some restriction conditions on both meshes and diffusion coefficients must be imposed, e.g., in [13] for MFD methods a set of constraint inequalities for the elements of the mass matrix of every mesh element is given. For a scheme without preserving monotonicity, it is easy to generate non-physical numerical solutions, especially on distorted meshes, such as negative temperature for thermal conduction problem. In order to avoid such numerical oscillation, monotonicity is regarded as an indispensable requirement in constructing discrete schemes.

In order to guarantee monotonicity without any severe condition, some nonlinear schemes [16,19,24] have been proposed. A nonlinear monotone FV scheme for highly anisotropic diffusion operators on unstructured triangular meshes has been proposed in [19] for parabolic equations with sufficiently small time steps. A nonlinear FV scheme for elliptic problems on triangular meshes has been proposed in [16], where a special choice of collocation points (i.e., cell centers) is introduced to prove the scheme being monotone on triangular meshes for strongly anisotropic and heterogeneous full tensor coefficients, but for general polygonal meshes some restrictions on diffusion tensor coefficients and meshes are also needed. A further developed monotone scheme for strongly anisotropic and heterogeneous tensor coefficients on star-shaped polygonal meshes has been proposed in [24], where it is unnecessary to choose a specific collocation points. To our knowledge, all of these existing monotone schemes are designed on conforming meshes.

In solving the radiation hydrodynamic problems arising from some applications such as inertial confinement fusion and nuclear reactor, it is important to guarantee that the direction of heat propagation is correct, i.e., heat should flow from higher temperature to lower temperature on distorted and possibly non-conforming meshes. Then it is essentially necessary to construct monotone diffusion scheme on general non-conforming meshes, which is just the aim of this paper. The nonlinear monotone scheme in [24] will be extended to non-conforming meshes, moreover a new method will be proposed to eliminate auxiliary vertex unknowns, in particular those defined at hanging nodes. This new method treats material discontinuities rigorously, and is adapted to almost arbitrary mesh geometry. The resulting finite volume scheme satisfies the following desirable properties:

- It is a cell-centered conservative scheme.
- It is monotone on almost arbitrary meshes including non-conforming cells.