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Generalized convolution-type singular integral equations

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ABSTRACT

In this paper, we study one class of generalized convolution-type singular integral equations in class {0}. Such equations are turned into complete singular integral equations with nodal points and further turned into boundary value problems for analytic function with discontinuous coefficients by Fourier transforms. For such equations, we will propose one method different from classical one and obtain the general solutions and their conditions of solvability in class {0}. Thus, this paper generalizes the theory of classical equations of convolution type.

1. Introduction

It is well-known that singular integral equations (SIEs) and boundary value problems (BVPs) for analytic functions are main branches of complex analysis and have a lot of applications, e.g., in elasticity theory, fluid dynamics, shell theory, underwater acoustics, quantum mechanics. The theory is well developed by many authors. Muskhelishvilli [1] studied SIE with Cauchy kernel as well as convolution kernel, especially the solvable Noether theory. Litvinchuc [2] extended the theory to the more general integral equations with convolution kernel and equations with Hilbert kernel in either closed curves or open arcs. Later, Du and Shen [3] further considered integral equations of convolution type with variable coefficients. Recently, Li and Ren [4] investigated some classes of convolution equations with harmonic singular operator, which can be translated into Riemann boundary value problems (RBVPs) with discontinuous coefficients via Fourier transforms and given the conditions of solvability and the explicit solutions.

The purpose of this paper is to extend further the theory to generalized convolution-type singular integral equations with Cauchy kernel. By applying the integral equation theory and the generalized resolvent kernel operator theory, we discuss the problem to find solutions for the equation in the normal type and the non-normal type cases. The general solutions and the conditions of solvability are obtained in class {0} by our method. This paper improves some results for Refs. [5–10].

2. Preliminaries

In this section, we give some classes of functions and their properties.

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Definition 2.1. Let $F(x)$ be a continuous function on $[-M, M]$. If there exists some positive real number B such that for any $x_1, x_2 \in [-M, M]$, the following condition

$$|F(x_1) - F(x_2)| \le B|x_1 - x_2|^\sigma \quad (0 < \sigma \le 1)$$

holds, we say that $F(x) \in H^\sigma$.

For simplification, we denote $F(x) \in H$.

Definition 2.2. Assume that $F(x)$ is a continuous on the whole real domain $\mathbb{R}$. We say that $F(x) \in \hat{H}$ if the following conditions are fulfilled:

(1) $F(x) \in H$ on $[-M, M]$ for any sufficient large positive number M.
(2) $|F(x_1) - F(x_2)| \le A|\frac{1}{x_1} - \frac{1}{x_2}|$ for any $|x_j| > M (j = 1, 2)$ and some positive constant A.

Definition 2.3. Assume that $F(x)$ satisfies the following conditions:
(1) $F(x) \in \widehat{H}$. (2) $F(x) \in L^1(\mathbb{R})$, where $L^1(\mathbb{R}) = \{F(x) \mid \int_{\mathbb{R}} |F(x)|dx < +\infty\}$.
We say that $F(x)$ belongs to class $\{\{0\}\}$.
If a function $F(x)$ satisfies the Hölder condition on a neighborhood N_∞ of ∞, we denote as $F(x) \in H(N_\infty)$.

Definition 2.4 (see [11]). A function $f(t)$ belongs to class $\{0\}$, if its Fourier transform

$$\mathbb{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{ixt} dt$$

belongs to the class $\{\{0\}\}$, we denote as $\mathbb{F}[f(t)] = F(x)$.

For any functions $h(t)$, $\varphi(t) \in \{0\}$, their convolution is denoted by

$$h * \varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} h(t-s)\varphi(s)ds, \quad t \in \mathbb{R}.$$

We also introduce the operator T of Cauchy principal value integral as

$$T\varphi(t) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(s)}{s-t} ds.$$

Lemma 2.1. *If $\varphi(t) \in \{0\}$, then $\mathbb{F}[T\varphi(t)] = -\Phi(x)\mathrm{sgn}(x)$, where $\Phi(x) = \mathbb{F}[\varphi(t)]$.*

Proof. Since

$$\mathbb{F}[T\varphi(t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(\tau)}{\tau - t} d\tau\right] e^{ixt} dt = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\frac{1}{\pi i} \int_{\mathbb{R}} \frac{e^{ixt}}{t-\tau} dt\right] \varphi(\tau) d\tau, \tag{2.1}$$

by extended Residue theorem, we have

$$\frac{1}{\pi i} \int_{\mathbb{R}} \frac{e^{ixt}}{t-\tau} dt = \begin{cases} e^{ix\tau}, & if \quad x > 0, \\ 0, & if \quad x = 0, \\ -e^{ix\tau}, & if \quad x < 0. \end{cases} \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\mathbb{F}[T\varphi(t)] = -\mathrm{sgn}(x) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi(t) e^{ixt} dt = -\mathrm{sgn}(x)\Phi(x). \tag{2.3}$$

Similarly $\mathbb{F}[T\varphi(-t)] = -\Phi(-x)\mathrm{sgn}(x)$. □

Lemma 2.2. *If $\varphi(t) \in \{0\}$, $\Phi(x) = \mathbb{F}[T\varphi(t)]$, then $\mathbb{F}[\mathrm{sgn}(t)\varphi(t)] = T\Phi(x)$.*

Proof. Since

$$\begin{aligned} \mathbb{F}[\mathrm{sgn}(t)\varphi(t)] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{sgn}(t)\varphi(t) e^{ixt} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^+} \varphi(t) e^{ixt} dt - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^-} \varphi(t) e^{ixt} dt \\ &= \Phi^+(x) + \Phi^-(x) \end{aligned}$$

and $\Phi(x) \in \{0\}$, $T\Phi(x) = \Phi^+(x) + \Phi^-(x)$, we have $\mathbb{F}[\mathrm{sgn}(t)\varphi(t)] = T\Phi(x)$,
where $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$. □

Lemma 2.3 (see [11]). *If $\varphi(t) \in \{0\}$, $\psi(t) \in \{0\}$, then $\varphi^* \psi(t) \in \{0\}$, $\mathbb{F}[\varphi * \psi(t)] = \Phi(x)\Psi(x)$.*