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Error estimates on a finite volume method for diffusion problems with interface on rectangular grids

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ABSTRACT

The finite volume methods are frequently employed in the discretization of diffusion problems with interface. In this paper, we firstly present a vertex-centered MACH-like finite volume method for solving stationary diffusion problems with strong discontinuity and multiple material cells on the Eulerian quadrilateral grids. This method is motivated by Frese [No. AMRC-R-874, Mission Research Corp., Albuquerque, NM, 1987]. Then, the local truncation error and global error estimates of the degenerate five-point MACH-like scheme are derived by introducing some new techniques. Especially under some assumptions, we prove that this scheme can reach the asymptotic optimal error estimate $O(h^2|\ln h|)$ in the maximum norm. Finally, numerical experiments verify theoretical results.

1. Introduction

Diffusion problems with interface are widely applied in multi-fluid hydrodynamic, fluid-solid coupling mechanics and many other scientific and engineering computation fields. The finite volume method (FVM), which presents local conservation and obvious advantage to handle physical models with complex characteristics very well, becomes an important discretization method for solving partial differential equations.

The finite volume methods (FVMs) based on Lagrangian and Eulerian grids are two commonly used methods for solving diffusion problems. The moving interface can be accurately described as we use the former, but the calculation is hard to execute on highly distorted grids. There are a lot of researches interested in these FVMs, e.g. [1–6]. An advantage of the FVMs in the Eulerian frame is the reasonable shape of the computational grids, such as the uniform grids. But for the diffusion problems with strong discontinuity on the internal interface, the difficulty lies in dealing with the cells involving multiple material properties [7]. Many researchers investigated this kind of FVMs, e.g. [8–19]. Ewing et al. [8] and Zhu et al. [9] constructed an immersed finite volume element method (FVEM) on a uniform triangle grid, and presented the optimal error estimate in the energy norm. Shu et al. [10] and Nie et al. [11] established the superconvergence theory of a bilinear FVEM for diffusion problems with smooth coefficients on the rectangular grids. A recovery-based posterior error estimator

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for FVMs to solve elliptic interface problems is presented in [12]. There also have a lot of works on the quadrilateral grids. For example, [13] and [14] focused on the two-dimensional (2D) and three-dimensional (3D) problems with discontinuous fluxes across the interface, and constructed a linear immersed FVEM. A series of FVMs with different average methods for diffusion problems under homogeneous jump conditions is constructed, and numerical experiments were carried on to confirm the approximation order of these methods in [15]. In [16] a kind of FVM by using a decomposition technique is derived. The paper [17] constructed a fourth-order compact FVM and derived some high accuracy post-processing formulas. Lv and Li [18,19] proved the optimal L^2 error estimate for bilinear and biquadratic FVMs with smooth coefficients under a mesh restriction of h^2-parallelogram, respectively. In addition, there are many works for the error estimates of finite element method to solve diffusion problems with smooth coefficients, for example, [20] derived the optimal L^2 error estimate of linear finite element scheme for the diffusion problem with nonlocal boundary. However, there have been few strict theoretical analyses of the global error estimation for Eulerian FVMs to solve diffusion problems with strong discontinuity and multiple material cells.

In the late 1980s, Frese presented a finite volume method for diffusion equations in 2D magnetohydrodynamic problems on the quadrilateral grids [21]. The corresponding software packages named MACH2 and MACH3 for 2D and 3D problems have been successfully developed, respectively, by Philips Lab/WSP, Kirtland Air Force Base [21,22], and widely used in the numerical simulations of liner implosion system, plasma thrusters and so on (see, e.g. [23–27]). For simplicity of presentation, we denote this finite volume method as MACH FVM. However, to our knowledge, we have not found the strict error theories of it for diffusion problems with strong discontinuity and multiple material cells on the Eulerian grids, which urges us to study it.

In this paper, we present a vertex-centered MACH-like FVM on the quadrilateral grids for stationary diffusion problems with interface. In particular, for the square grids, this kind of nine-point scheme is degenerated into a five-point scheme. It is worth pointing out that many classical nine-point FVMs (see, for example, [28]) are always degenerated into a five-point stencil which looks like "+", while the stencil of the five-point MACH-like scheme looks like "×". Therefore, this adds an extra difficulty for error estimates. The other important work of this paper is that we present the strict theoretical analysis for the five-point MACH-like scheme. It is divided into two parts. On the one hand, we discuss the local truncation error. The main difficulty results from the discontinuous coefficients of the interface. By the homogeneous jump conditions on the interface and Taylor expansions, we derive that the local truncation error of the interior nodes adjacent to the interface is $O(h)$. Furthermore, we get the local truncation error $O(h^2)$ under the Assumption **I** (i.e., we use harmonic average method for the diffusion coefficients and the second derivative function with respect to the tangent direction of the exact solution is equal to zero on the interface). On the other hand, we focus on the global error estimates. Firstly, the error difference equations are decomposed into two relatively simple ones. Then, by using the discrete sine transform [29,30] and combining with some analytical techniques, we convert these 2D difference equations into two kinds of one-dimensional (1D) difference equations, and the estimates of these difference equations are deduced. Hence, we demonstrate that the global error estimation of the five-point scheme is $O(h|\ln h|)$ in the maximum norm. Furthermore, under the Assumption **I**, we obtain the asymptotic optimal error estimate $O(h^2|\ln h|)$. In addition, we investigate the approximation of the five-point MACH-like scheme by several typical numerical examples. Numerical results are carried on to confirm the theoretical ones.

The paper is organized as follows. In Section 2, we describe the construction of the MACH-like finite volume method on an arbitrary quadrilateral grid, and present a five-point MACH-like scheme for special. Section 3 presents the local truncation error for the five-point scheme. Then the global error estimation in the maximum norm is derived in Section 4. After that, the accuracy of the five-point MACH-like scheme is verified numerically. In Section 6 we draw some conclusions on our works.

2. Model problem and finite volume scheme

We consider the following interface problem

$$\begin{cases} -\nabla\cdot(\kappa\nabla u)+u=f, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega\in\mathbb{R}^2$ is a bounded polygonal domain with boundary $\partial\Omega$, f is a given function and the diffusion coefficient κ is positive and piecewise constant on polygonal subdomains of Ω with possible large discontinuities across subdomain boundaries which is simply interface for short. Let $\kappa=\kappa_i>0$ in D_i, for $i=1,2,\dots,J$. Here $\{D_i\}_{i=1}^J$ is a partition of Ω, where D_i is an open polygonal subdomain.

Denote $\Gamma_{i,j}=\bar{D}_i\cap\bar{D}_j(i\neq j)$ and $\Gamma=\cup_{i,j=1}^J\Gamma_{i,j}$. We assume that

$$[u]=\left[\kappa\frac{\partial u}{\partial\vec{n}}\right]=0,\quad \text{on } \Gamma, \tag{2.2}$$

where $\vec{n}$ is the unit outward normal vector on Γ and $[\zeta](\zeta=u,\kappa\frac{\partial u}{\partial\vec{n}})$ denotes the difference of the right and left limits of ζ at any point of Γ.

Let $\mathcal{Q}_h$ be a structured quadrilateral partition of Ω, and $X=\{X_{i,j}=(x_i,y_j),\ i=0,1,\dots,N_x, j=0,1,\dots,N_y\}$, where N_x, N_y are given positive integers.