Applied Mathematics and Computation 311 (2017) 353–360

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Parameterized outer estimation of *AE*-solution sets to parametric interval linear systems

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ARTICLE INFO

MSC:
65G40
15A06

Keywords:
Linear equations
Dependent interval parameters
AE-solution set
Outer estimation

ABSTRACT

We consider linear algebraic equations, where the elements of the matrix and of the right-hand side vector are linear functions of interval parameters, and their parametric *AE*-solution sets, which are defined by applying universal and existential quantifiers to the interval parameters. Usually, interval methods find numerical interval vector that contains an *AE*-solution set.

In this work we propose a method that generates an outer estimate of a parametric *AE*-solution set in form of a linear parametric interval function, called parameterized outer solution (p-solution). Parameterized outer solution is proposed for the parametric united solution set in Kolev (2014) and takes precedence over the classical interval solution enclosure when the latter is part of other problems involving the same parameters. The method we present generalizes the method from Kolev (2016) for parametric *AE*-solution sets. It is also a parameterized analogue of a method from Popova and Hladík (2013) and produces the same interval enclosure as the method from the last reference.

1. Introduction

Denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real vectors with n components and the set of real $m \times n$ matrices, respectively. Vectors are considered as one-column matrices. A real compact interval is $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. We consider systems of linear algebraic equations having linear uncertainty structure

$$\begin{aligned} &A(p)x \; = \; a(p), \quad p \in \mathbf{p}, \\ &A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, \qquad a(p) := a_0 + \sum_{k=1}^{K} p_k a_k, \end{aligned} \tag{1}$$

where $A_k \in \mathbb{R}^{n\times n}$, $a_k \in \mathbb{R}^n$, $k = 0, \dots, K$, and the parameters $p = (p_1, \dots, p_K)^\top$ are considered to be uncertain and varying within given intervals $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_K)^\top$.

We consider parametric *AE*-solution sets of system (1), which are defined by

$$\begin{aligned} \Sigma^p_{AE} &= \Sigma_{AE}(A(p), b(p), \mathbf{p}) \\ &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p)x = b(p))\}, \end{aligned} \tag{2}$$

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http://dx.doi.org/10.1016/j.amc.2017.05.042
0096-3003/

where $\mathcal{A}$ and $\mathcal{E}$ are sets of indexes such that $\mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}$, $\mathcal{A} \cap \mathcal{E} = \emptyset$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$. Particular index sets $\mathcal{A}, \mathcal{E}$ are associated to each particular *AE*-solution set. Some of the most studied and with more practical interest *AE*-solution sets are: the (parametric) united solution set

$$\Sigma^p_{\text{uni}} = \Sigma_{\text{uni}}(A(p), b(p), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma_{AE}(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\},$$

and the (parametric) controllable solution set

$$\Sigma_{AE}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

For interpretation and applications of *AE*-solution sets see, e.g., [1].

Usually, when finding outer interval enclosure of a solution set, interval methods generate a numerical interval vector that contains the particular *AE*-solution set. For inner estimation of parametric *AE*-solution sets see, e.g., [2]. A new type of enclosure, called *parameterized* or *p-solution*, providing outer estimate of the parametric united solution set is proposed in [3]. The proposed p-solution is in form of a linear parametric interval function

$$x(p, l) = Lp + l, \quad p \in \mathbf{p},\ l \in \mathbf{l},$$

where L is a real $n \times K$ matrix and $\mathbf{l}$ is an n-dimensional interval vector. The parameterized solution has the property

$$\Sigma^p_{uni} \subseteq x(\mathbf{p}, \mathbf{l}),$$

where $x(\mathbf{p}, \mathbf{l})$ is the interval hull of $x(p, l)$ over $p \in \mathbf{p}$, $l \in \mathbf{l}$. For a nonempty and bounded set $\Sigma \subset \mathbb{R}^n$, its interval hull is defined by

$$\square\Sigma := \bigcap\{\mathbf{x} \in \mathbb{IR}^n \mid \Sigma \subseteq \mathbf{x}\}.$$

Some iterative methods for determining $x(p, l)$ of the parametric united solution set are proposed in [3,4]. In order to improve their computational efficiency, a direct method for determining the p-solution $x(p, l)$ is proposed in [5]. A numerical example demonstrates in [4] that the parameterized solution is rather promising in solving some global optimization problems where the parametric linear system (1) is involved as an equality constraint.

In this work we generalize the direct method, developed in [5] for the parametric united solution set, to arbitrary parametric *AE*-solution sets and propose a method that generates an outer solution enclosure in a form, which is not an interval vector but a linear parametric interval function $x(p, l)$. The structure of the paper is as follows. Section 2 contains notation and basic facts about the arithmetic on proper and improper intervals [6], which will be used to simplify the proof of Theorem 6 in Section 3, as well as various known results that are used as a background for the derivation of the parameterized *AE*-solution. The parameterized *AE*-solution and its interval enclosure property are proven in Section 3. A numerical algorithm implementing the parameterized *AE*-method is presented in Section 4 along with some numerical examples. The paper ends by some conclusions.

2. Preliminaries

For $\mathbf{a} = [\underline{a}, \overline{a}]$, define the mid-point $\check{a} := (\underline{a} + \overline{a})/2$, the radius $\hat{a} := (\overline{a} - \underline{a})/2$ and the absolute value (magnitude) $|\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\}$. These functions are applied to interval vectors and matrices componentwise. Inequalities are understood componentwise. The spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\varrho(A)$. The identity matrix of appropriate dimension is denoted by I.

In order to simplify the presentation, in Section 3 we use the arithmetic on proper and improper intervals [6], called Kaucher complete arithmetic, and its properties; see also [7]. The set of proper intervals $\mathbb{IR}$ is extended in [6] by the set $\overline{\mathbb{IR}} := \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}, a_1 \geq a_2\}$ of *improper* intervals obtaining thus the set $\mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}\}$ of all ordered couples of real numbers, called also generalized intervals. The conventional interval arithmetic operations, lattice operations intersection ($\cap$) and union ($\cup$), order relations, and other interval functions are isomorphically embedded into the whole set $\mathbb{IR} \bigcup \overline{\mathbb{IR}}$ [6].

An element-to-element symmetry between proper and improper intervals is expressed by the "dual" operator, $\text{dual}(\mathbf{a}) := [a_2, a_1]$ for $\mathbf{a} = [a_1, a_2] \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$. The operator `dual` is applied componentwise to vectors and matrices. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$

$$\begin{gathered}\text{dual}(\text{dual}(\mathbf{a})) = \mathbf{a}, \qquad \text{dual}(\mathbf{a} \circ \mathbf{b}) = \text{dual}(\mathbf{a}) \circ \text{dual}(\mathbf{b}),\ \ \circ \in \{+, -, \times, /\}, \\ \mathbf{a} \subseteq \mathbf{b} \Leftrightarrow \text{dual}(\mathbf{a}) \supseteq \text{dual}(\mathbf{b}). \end{gathered} \tag{3}$$

The generalized interval arithmetic structure possesses group properties with respect to addition and multiplication operations. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$, $0 \notin \mathbf{b}$

$$\mathbf{a} - \text{dual}(\mathbf{a}) = 0, \qquad \mathbf{b}/\text{dual}(\mathbf{b}) = 1. \tag{4}$$