Applied Mathematics and Computation 311 (2017) 361–374

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A second order Crank–Nicolson scheme for fractional Cattaneo equation based on new fractional derivative

Zhengguang Liu, Aijie Cheng*, Xiaoli Li

School of Mathematics, Shandong University, Jinan, Shandong 250100, China

ARTICLE INFO

MSC:
65M06
65M12
65M15
26A33

Keywords:
Second order
New fractional derivative
Crank–Nicolson
Cattaneo equation
Finite difference

ABSTRACT

Recently Caputo and Fabrizio introduce a new derivative with fractional order which has the ability to describe the material heterogeneities and the fluctuations of different scales. In this article, a Crank–Nicolson finite difference scheme to solve fractional Cattaneo equation based on the new fractional derivative is introduced and analyzed. Some a priori estimates of discrete $L^\infty(L^2)$ errors with optimal order of convergence rate $O(\tau^2+h^2)$ are established on uniform partition. Moreover, the applicability and accuracy of the scheme are demonstrated by numerical experiments to support our theoretical analysis.

1. Introductions

Differential equations involving derivatives of fractional order have shown to be adequate models for various physical phenomena in areas like rheology, damping laws, diffusion processes, etc. [1–15], which gives attractive applications as a new modeling tool in a variety of scientific and engineering fields. Numerical methods and theory of solutions for fractional differential equations have been studied extensively by many researchers which mainly cover finite element methods [1–5], mixed finite element methods [16], finite difference methods [7,17–23], finite volume (element) methods [24,25], (local) discontinuous Galerkin (L)DG methods [26], spectral methods [27,28] and so on.

Caputo and Fabrizio [29] suggest a new definition of fractional derivative, which assumes two different representations for the temporal and spatial variable. The new definition can portray substance heterogeneities and configurations with different scales, which noticeably cannot be managing with the renowned local theories. In [30], the authors present an alternative representation of the diffusion equation and the diffusion-advection equation using the fractional calculus approach, the spatial-time derivatives are approximated using the fractional definition recently introduced by Caputo and Fabrizio in the range $\alpha \in (0, 2]$. They prove that when $\alpha \in (1, 2]$, the concentration exhibits the Markovian Lévy flights and the superdiffusion phenomena. From [30], one can see that this derivative possesses very interesting properties, for instance, the possibility to describe fluctuations and structures with different scales. Furthermore, this definition allows for the description of mechanical properties related with damage, fatigue, material heterogeneities and structures at different scales. An additional appliance is in the study of the macroscopic behaviors of some materials, associated with non-local communications between atoms, which are recognized in important of the properties of material. Properties and applications of this new fractional derivative are reviewed in detail in the papers [31–33]. Atangana [31] introduces the application to

* Corresponding author.
E-mail addresses: liuzhgsdu@yahoo.com (Z. Liu), aijie@sdu.edu.cn (A. Cheng), xiaolisdu@163.com (X. Li).

http://dx.doi.org/10.1016/j.amc.2017.05.032
0096-3003/

nonlinear Fishers reaction-diffusion equation based on the new fractional derivative. He [32] also analyzes the extension of the resistance, inductance, capacitance electrical circuit to this fractional derivative without singular kernel. A numerical solution for the model of resistance, inductance, capacitance(RLC) circuit via the fractional derivative without singular kernel is considered by Atangana [33].

The time fractional Cattaneo equation has recently been considered by many researchers [34–39]. For instance, Compte and Metzler [35] first extend the Cattaneo equation to anomalous transport processes and three meaningful generalized Cattaneo equation. Lewandowska [36] applied the generalized Cattaneo equation to investigate the transport process of electrolytes in sub-diffusive media and obtained the formula of electrochemical subdiffusive impedance of a spatially limited sample for large pulsation of electric field. Lewandowska [37] utilized two different hyperbolic Cattaneo equations with the fractional time derivatives to model the subdiffusion impedance of a homogeneous sample of finite thickness. In numerical analysis of the equation, Vong et al. [34] consider a high-order difference scheme for the generalized Cattaneo equation. Qi and Jiang [38] derived the exact solution of the space-time fractional Cattaneo model using the joint Laplace and Fourier transform. Compact Crank–Nicolson schemes for a class of fractional Cattaneo equation in inhomogeneous medium are introduced by Zhao and Sun [39].

In this article, we consider the following time fractional Cattaneo equation:

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t), \tag{1}$$

where $(x,t)\in\Omega=[0,L]\times[0,T]$, $1<\alpha<2$, $f\in C[0,T]$, with the initial conditions

$$u(x,0)=\phi(x),\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=\psi(x),\quad 0\le x\le L, \tag{2}$$

and boundary conditions

$$u(0,t)=u(L,t)=0,\quad t>0. \tag{3}$$

Our target is to present a Crank–Nicolson finite difference method to solve a time-fractional Cattaneo equation based on the new fractional derivative. The unconditionally stable result, which just depends on initial value and source item, is derived. Some a priori estimates of discrete $L^{\infty}(L^2)$ errors with optimal order of convergence $O(\tau^2+h^2)$ are obtained on uniform partition.

The paper is organized as follows. In Section 2, we give the definitions and some notations. We introduce a Crank–Nicolson finite difference scheme for fractional Cattaneo equation in Section 3. Then in Section 4, we present the analysis of stability and error estimates for the presented method. In Section 5, some numerical experiments using the second order finite difference scheme are carried out.

2. Some notations and definitions

Firstly, we give some definitions which are used in the following analysis.

Let us recall the usual Caputo fractional time derivative of order α, given by

$${}_a^C D_t^{\alpha}u(x,t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^{(n)}(x,s)ds,\quad n-1<\alpha<n.$$

By changing the kernel $(t-s)^{-\alpha}$ with the function $\exp(-\alpha\frac{t-s}{1-\alpha})$ and $\frac{1}{\Gamma(1-\alpha)}$ with $\frac{M(\alpha)}{1-\alpha}$, Caputo and Fabrizio give the following new definition of fractional time derivative.

Definition 1 [29]**.** Let $u(\cdot,t)\in H^1(a,b)$, $b>a$, $\alpha\in(0,1)$, then the new Caputo derivative of fractional order is defined as:

$$D_t^{\alpha}u(x,t)=\frac{M(\alpha)}{1-\alpha}\int_a^t u'(x,s)\exp\left[-\alpha\frac{t-s}{1-\alpha}\right]ds,$$

where $M(\alpha)$ is a normalization function such that $M(0)=M(1)=1$. But if the function u does not belong to $H^1(a,b)$ then the derivative can be reformulated as

$$D_t^{\alpha}u(x,t)=\frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (u(x,t)-u(x,s))\exp\left[-\alpha\frac{t-s}{1-\alpha}\right]ds.$$

Definition 2 [31]**.** Losada and Nieto proposed that the new Caputo derivative of order $0<\alpha<1$ can be reformulated as

$${}_0^{CF}D_t^{\alpha}u(x,t)=\frac{1}{1-\alpha}\int_0^t u'(x,s)\exp\left[-\alpha\frac{t-s}{1-\alpha}\right]ds.$$

Definition 3 [31]**.** Let $u(\cdot,t)\in H^1(a,b)$, if $n\ge 1$, and $\alpha\in[0,1]$ the fractional time derivative ${}_0^{CF}D_t^{\alpha}u(x,t)$ of order $(n+\alpha)$ is defined by

$${}_0^{CF}D_t^{n+\alpha}u(x,t)={}_0^{CF}D_t^{\alpha}\left({}_0^{CF}D_t^{n}u(x,t)\right).$$