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New results on the constants in some inequalities for the Navier–Stokes quadratic nonlinearity

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ABSTRACT

We give fully explicit upper and lower bounds for the constants in two known inequalities related to the quadratic nonlinearity of the incompressible (Euler or) Navier–Stokes equations on the torus $\mathbf{T}^d$. These inequalities are "tame" generalizations (in the sense of Nash–Moser) of the ones analyzed in the previous works (Morosi and Pizzocchero (2013) [6]).

1. Introduction

Let us consider the homogeneous incompressible Navier–Stokes (NS) equations on a torus $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$ of arbitrary dimension; the nonlinear part of these equations is governed by the bilinear map $\mathscr{P}$ sending two sufficiently regular vector fields $v, w : \mathbf{T}^d \to \mathbf{R}^d$ into

$$\mathscr{P}(v, w) := \mathfrak{L}(v \cdot \partial w) \ . \tag{1.1}$$

In the above $v \cdot \partial w : \mathbf{T}^d \to \mathbf{R}^d$ is the vector field of components $(v \cdot \partial w)_s := \sum_{r=1}^d v_r \partial_r w_s$ and $\mathfrak{L}$ is the Leray projection onto the space of divergence free vector fields (see Section 2 for more details). Of course the NS equations read

$$\frac{\partial u}{\partial t} = \nu \Delta u - \mathscr{P}(u, u) + f, \tag{1.2}$$

where: $u = u(x, t)$ is the divergence free velocity field, depending on $x \in \mathbf{T}^d$ and on time t; $\nu \geq 0$ is the kinematic viscosity, Δ is the Laplacian of $\mathbf{T}^d$; $f = f(x, t)$ is the (Leray projected) external force per unit mass. In the inviscid case $\nu = 0$, (1.2) become the Euler equations.

In this paper, we focus the attention on certain inequalities fulfilled by $\mathscr{P}$ in the framework of Sobolev spaces. For any real n, we denote with $\mathbb{H}^n_{\Sigma 0}$ the Sobolev space formed by the (distributional) vector fields v on $\mathbf{T}^d$ with vanishing divergence and mean, such that $\sqrt{-\Delta}^n v$ is in L^2; this carries the inner product $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w\rangle_{L^2}$ and the norm $\|v\|_n := \sqrt{\langle v|v\rangle_n}$ (see the forthcoming Eqs. (2.8) and (2.9)). Let p, n be real numbers; it is known that

$$n > d/2, \ \ v \in \mathbb{H}^n_{\Sigma 0}, \ \ w \in \mathbb{H}^{n+1}_{\Sigma 0} \quad \Rightarrow \quad \mathscr{P}(v, w) \in \mathbb{H}^n_{\Sigma 0} \tag{1.3}$$

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and that there are positive real constants K_n, G_n, K_{pn}, G_{pn} such that:

$$\|\mathscr{P}(v,w)\|_n \leqslant K_n \|v\|_n \|w\|_{n+1} \qquad \text{for}, n > d/2, v \in \mathbb{H}^n_{\Sigma 0}, w \in \mathbb{H}^{n+1}_{\Sigma 0}\ , \tag{1.4}$$

$$|\langle \mathscr{P}(v,w)|w\rangle_n| \leqslant G_n \|v\|_n \|w\|^2_n \qquad \text{for}, n > d/2+1, v \in \mathbb{H}^n_{\Sigma 0}, w \in \mathbb{H}^{n+1}_{\Sigma 0}\ , \tag{1.5}$$

$$\|\mathscr{P}(v,w)\|_p \leqslant \frac{1}{2} K_{pn}(\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}) \qquad \text{for}, p \geqslant n > d/2, v \in \mathbb{H}^p_{\Sigma 0}, w \in \mathbb{H}^{p+1}_{\Sigma 0}\ , \tag{1.6}$$

$$|\langle \mathscr{P}(v,w)|w\rangle_p| \leqslant \frac{1}{2} G_{pn}(\|v\|_p \|w\|_n + \|v\|_n \|w\|_p)\|w\|_p \qquad \text{for}, p \geqslant n > d/2+1, v \in \mathbb{H}^p_{\Sigma 0}, w \in \mathbb{H}^{p+1}_{\Sigma 0}. \tag{1.7}$$

Statements (1.3) and (1.4) indicate that $\mathscr{P}$ maps continuously $\mathbb{H}^n_{\Sigma 0} \times \mathbb{H}^{n+1}_{\Sigma 0}$ to $\mathbb{H}^n_{\Sigma 0}$ if $n > d/2$. Eq. (1.6) with $p = n$ implies Eq. (1.4), with $K_n := K_{nn}$; similarly, (1.7) with $p = n$ gives (1.5) with $G_n := G_{nn}$.

Eq. (1.4) is closely related to the basic norm inequalities about multiplication in Sobolev spaces, and (1.5) is due to Kato [5]; for these reasons, in [11,12] inequalities (1.4) and (1.5) are referred to, respectively, as the “basic” and “Kato” inequalities for $\mathscr{P}$[1]. Eqs. (1.6) and (1.7) are tame refinements of (1.4) and (1.5) (in the general sense given to tameness in studies on the NashMoser implicit function theorem [4]). We remark that inequalities very similar to (1.7) are used by Temam in [16], Beale–Kato–Majda in [1] and Robinson–Sadowski–Silva in the recent work [15].

From here to the end of the paper we intend that K_n, G_n, K_{pn}, G_{pn} are, respectively, the *sharp* constants in (1.4), (1.5), (1.6), and (1.7) (i.e., the minimum constants fulfilling these inequalities). In the previous papers [11,12], explicit upper and lower bounds were provided for K_n and G_n. In the present work, we generalize the cited results deriving upper and lower bounds for K_{pn} and G_{pn}, for all real p, n as in Eqs. (1.6) and (1.7). Our derivations of the upper bounds also give, as byproducts, simple and self-consistent proofs of the related inequalities; the proposed approach follows ideas from Temam [16] and Constantin–Foias [3], making them more quantitative. The lower bounds are obtained substituting suitable trial vector fields in Eqs. (1.6) and (1.7).

The relevance of a quantitative information on the constants K_{pn}, G_{pn} is pointed out, e.g., in [14]. In the cited work, inequalities (1.4) – (1.7) and the constants therein are used to give bounds on the exact C^∞ solution of the NS Cauchy problem with smooth initial data (including the Euler case $\nu = 0$) via the *a posteriori* analysis of an approximate solution; these estimates concern the interval of existence of the exact solution and its Sobolev distance of any order from the approximate solution. Paper [14] uses systematically the known fact that the space of C^∞ vector fields on $\mathbf{T}^d$ with vanishing divergence and mean coincides with $\cap_{p\in\mathbf{R}}\mathbb{H}^p_{\Sigma 0}$; the tame structure of inequality (1.7) is essential for an efficient implementation of the a posteriori analysis since, after fixing a basic order $n > d/2+1$, it induces simple estimates in terms of the Sobolev norms of arbitrary order $p \geq n$. The setting of [14] is in fact a C^∞ variant of the framework introduced in [10] (and inspired by Chernyshenko et al. [2]), where the exact and approximate NS solutions live in a Sobolev space of a given finite order, and the a posteriori analysis is based only on inequalities (1.4) and (1.5). For some applications of the general schemes of [10,14], in addition to these papers we wish to mention [7,8,13].

Organization and main results of the paper. The present paper has an extended arXiv version [9], giving additional details on the most technical aspects of our estimates. Section 2 of the present work reviews some basic notations and presents a number of elementary facts about the bilinear map $\mathscr{P}$. The subsequent Sections 3 and 4 present our upper bounds K^+_{pn}, G^+_{pn} for the sharp constants (1.6) and (1.7), respectively; these are described by Theorems 3.3, 4.4 and have the form

$$K^+_{pn} = \frac{1}{(2\pi)^{d/2}}\sqrt{\sup_{k\in\mathbf{Z}^d\setminus\{0\}} \mathcal{K}_{pn}(k)}\ , \qquad G^+_{pn} = \frac{1}{(2\pi)^{d/2}}\sqrt{\sup_{k\in\mathbf{Z}^d\setminus\{0\}} \mathcal{G}_{pn}(k)} \tag{1.8}$$

where $\mathcal{K}_{pn}, \mathcal{G}_{pn} : \mathbf{Z}^d \setminus \{0\} \to [0, +\infty)$ are explicitly given, bounded functions. For each k, $\mathcal{K}_{pn}(k)$ and $\mathcal{G}_{pn}(k)$ are infinite (zeta-type) sums over the lattice $\mathbf{Z}^d$ or, to be precise, on $\mathbf{Z}^d\setminus\{0, k\}$: see Eqs. (3.2), (3.9), (4.3), and (4.11). Sections 3 and 4 also propose some elementary upper bounds on the sups in Eq. (1.8); these imply upper bounds $K^{\langle+\rangle}_{pn}$, $G^{\langle+\rangle}_{pn}$ for K_{pn} and G_{pn}, much rougher than K^+_{pn} and G^+_{pn}.

In Section 5, we sketch a procedure, suitable for computer implementation, to approximate accurately from above the functions $\mathcal{K}_{pn}, \mathcal{G}_{pn}$ and their sups; this procedure is described in full detail in the arXiv version [9]. The basic idea is to approximate the infinite sums $\mathcal{K}_{pn}(k), \mathcal{G}_{pn}(k)$ with finite sums over the integer points of suitable balls, giving accurate reminder estimates; in the same spirit, the sups of $\mathcal{K}_{pn}$ and $\mathcal{G}_{pn}$ over $\mathbf{Z}^d$ are approximated with sups over the integer points of a ball, giving again error estimates. This construction finally produces precise upper approximants $K^{(+)}_{pn}$, $G^{(+)}_{pn}$ for K^+_{pn}, G^+_{pn}. In [9], after developing the theoretical basis of the above approximation technique we describe its actual implementation in

[1] Due to a remark of [11], we could write inequality (1.5) and its extension (1.7) using, in place of $\mathscr{P}(v,w) = \mathfrak{L}(v\cdot w)$, the vector field (with non zero divergence) $v \cdot \partial w$. The cited reference considers the Sobolev space $\mathbb{H}^n_0$ of vector fields v on $\mathbf{T}^d$ with vanishing mean and $\sqrt{-\Delta}^n v$ in L^2, with the inner product $\langle v|w\rangle_n := \langle \sqrt{-\Delta}^n v|\sqrt{-\Delta}^n\rangle_{L^2}$; for any $n > d/2$ and $v \in \mathbb{H}^n_{\Sigma 0}$, $w \in \mathbb{H}^{n+1}_{\Sigma 0}$ one has $v\cdot\partial w \in \mathbb{H}^n_0$, $\mathscr{P}(v,w) \in \mathbb{H}^n_{\Sigma 0}$ and $\langle \mathscr{P}(v,w)|w\rangle_n = \langle v\cdot\partial w|w\rangle_n$. However, these considerations will play no role in the present paper.