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Coupon coloring of cographs

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ABSTRACT

Coupon coloring is a new coloring which has many applications. A k-coupon coloring of a graph G is a k-coloring of G by colors $[k]=\{1,2,\dots,k\}$ such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum integer k for which a k-coupon coloring exists is called the coupon coloring number of G, and it is denoted by $\chi_c(G)$. In this paper, we studied the coupon coloring of cographs, which are graphs that can be generated from the single vertex graph K_1 by complementation and disjoint union, and have applications in many interesting problems. We use the cotree representation of a cograph to give a polynomial time algorithm to color the vertices of a cograph, and then prove that this coloring is a coupon coloring with maximum colors, hence get the coupon coloring numbers of the cograph.

1. Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [2].

In graph theory, there are lots of colorings which have been well studied so far, such as the classical vertex coloring and edge coloring, list coloring, star coloring and acyclic coloring, etc. In recent years, the rainbow connection and the rainbow vertex-connection are new hot topics in graph colorings (see surveys [16,19], and recent papers [12,13,15,17,18]). Coupon coloring is a new vertex coloring which was introduced in [4].

Let G be a graph without isolated vertices. A k-vertex coloring, or simply a k-coloring of G is a mapping C from $V(G)$ to $[k]=\{1,2,\dots,k\}$. A k-coupon coloring of G is a k-coloring of G such that the neighborhood of every vertex of G contains vertices of all colors from $[k]$. The maximum integer k for which a k-coupon coloring exists is called the coupon coloring number of G, and it is denoted by $\chi_c(G)$. The coupon coloring number of a graph G without isolated vertices is well-defined, since every vertex can be assigned the same color. For a graph G with isolated vertices, set $\chi_c(G)=0$.

Coupon coloring is an interesting coloring, not only because of its theoretical value, but also because it can be used in the network science and some other related fields. It is very useful in large multi-robet networks [1,3]. On the other hand, the coupon coloring problem for hypercubes is closely related to the problems in coding theory, and for $k=2$ the coupon coloring problem is also equivalent to the well-studied Property B of hypergraphs [7]. In [4], Chen et al. showed that for every d-regular graphs G, $\chi_c(G)\geq(1-o(1))\frac{d}{\log d}$ as $d\to\infty$, and the proportion of d-regular graphs G for which $\chi_c(G)\leq(1+o(1))\frac{d}{\log d}$ tends to 1 as $|V(G)|\to\infty$. They also determined the bounds of the coupon coloring number of

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cubes. In [22], Shi et al. determined the coupon coloring numbers of complete graphs, complete k-partite graphs, wheels, cycles, unicyclic graphs, bicyclic graphs and generalized Θ-graphs.

Cograph is a graph that can be generated from the single vertex graph K_1 by complementation and disjoint union. It was discovered independently by Jung [11], Lerchs [14], Seinsche [20], Sumner [23], etc., since the 1970s. It has applications in many problems, such as examination scheduling problems [21], automatic clustering of index terms [9], and in the recognition of read-once functions [8].

In this paper, we consider the coupon coloring of cographs. In Section 2, we give some basic concepts of coupon colorings. And in Section 3 we introduce cotrees which is one to one corresponding to cographs. At last, in Section 4, we give an algorithm to give a $\chi_c(G)$-coupon coloring for any cograph G, and then the coupon coupon coloring number $\chi_c(G)$ can also be got.

2. Coupon colorings

In this section, we state basic concepts of coupon coloring, which will be used in our proof.

By the definition of coupon coloring, we can easily get that

Lemma 2.1. *Let $\delta(G)$ be the minimum degree of the graph G, then $\chi_c(G) \le \delta(G)$.*

In a coupon coloring of G, for any vertex v in G, the color that v has must also appear in v's neighborhood. This means that

Lemma 2.2. *Any color in a coupon coloring must appear at least twice.*

Therefore, we have

Lemma 2.3. *Let G be a graph with n vertices, then $\chi_c(G) \le \lfloor \frac{n}{2} \rfloor$.*

In [22], the coupon coloring of some graphs are considered, such as complete graphs, complete multi-partite graphs, etc. They showed that

Lemma 2.4. *[22] Let G be a complete graph with n vertices. Then $\chi_c(G) = \lfloor \frac{n}{2} \rfloor$.*

Lemma 2.5. *[22] Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \ge 3$ and $n_1 \le n_2 \le \cdots \le n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $n = \sum_{i=1}^{k} n_i$. Then*

$$\chi_c(G) = \begin{cases} \lfloor \frac{n}{2} \rfloor & if\ s \ge \frac{n}{2}, \\ s & otherwise. \end{cases}$$

For a graph G and its induced subgraph H, we can get the relationship between their coupon coloring numbers as follows.

Lemma 2.6. *If H is an induced subgraph of G, then $\chi_c(G) \ge \chi_c(H)$.*

Proof. Suppose C is coupon coloring of H with $\chi_c(H)$ colors. Then for any vertex $v \in V(H)$, the neighborhood of v contains vertices of all colors from $[\chi_c(H)]$.

Color the vertex of G in the same way of H. Since H is an induced subgraph of G, for any vertex $v \in V(G)$, $v \in V(H)$ and $N_H(v) \subseteq N_G(v)$. So the neighborhood of v contains vertices of all colors from $[\chi_c(H)]$. This implies that C is a coupon coloring of G with $\chi_c(H)$ colors. Therefore $\chi_c(G) \ge \chi_c(H)$. □

3. Cographs and cotrees

In this section, we state the special structure of cographs and the concept of cotrees which is well studied because each cograph has a unique cotree representation.

Any cograph may be constructed using the following rules:

1. any single vertex graph is a cograph;
2. if G is a cograph, so is its complement graph $\overline{G}$;
3. if G and H are cographs, so is their disjoint union $G \cup H$.

Alternatively, instead of using the complement operation, we can use the join operation [5]. The join of graphs G and H, denoted by $G + H$, consists of forming the disjoint union $G \cup H$ and then adding an edge between every pair of a vertex from G and a vertex from H. It is easy to see that $G + H = \overline{\overline{G} \cup \overline{H}}$. Then, we can get another definition of cographs as follows.

Definition 1. The class of cographs is the smallest class of graphs containing the single vertex graph and closed under union and join composition.

Therefore, each cograph can be associated with several composition formulas using union and join operations. Such a formula can be written as a tree whose leaves are the vertices of the graph, and the internal nodes are labeled union or