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# On the numerical approximation for Fourier-type highly oscillatory integrals with Gauss-type quadrature rules[☆]

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ABSTRACT

In this paper, we present an improved numerical steepest descent method for the approximation of Fourier-type highly oscillatory integrals. Based on the previous numerical steepest descent method, the new method used the integrand information at endpoints and stationary points. The asymptotic order is given that is improved both for the case of stationary points and stationary points free. Several numerical examples are presented which show the high efficiency of the proposed method. Numerical results support our theoretical analyses.



## 1. Introduction

Fourier-type oscillatory integrals frequently appear in many science and engineering applications which can be written as

$$I[f]=\int_a^b f(x)e^{i\omega g(x)}dx,\quad \omega\gg 1, \tag{1.1}$$

where $\omega$ is the frequency parameter. The function $f$ and $g$ are usually called the amplitude and phase function respectively. When $\omega$ is large the numerical evaluation of the above integral will come across serious difficulties, since the integrand is highly oscillatory in that case. The classical quadrature rules will encounter huge challenge because of the requirement of a large number of quadrature points which is infeasible usually from a computational point of view.

In the past few decades, the numerical computation of highly oscillatory integrals has been attracted much attention by many scholars. Many effective methods have been proposed for computing various oscillatory integrals, such as asymptotic method [5,14,15,26], Levin-type method [17–19,22], Filon-type method [2,4,6,7,16,23,24,29–33], numerical steepest descent method [8,9,11–13,20,25,27,28], and so on. In 1928, Filon [6] presented a method for the computation of the highly oscillatory integral

$$\int_a^b f(x)\sin(\omega x)dx$$

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by using a piecewise interpolation to $f$. After the work of Filon, many papers on his feet appeared, the 2005 paper of Iserles and Nørsett [14] is an important work on Filon-type methods. In their work, the asymptotic expansion of $\int_0^1 f(x)e^{i\omega g(x)}dx$ in inverse powers of the frequency $\omega$ has been studied. Then they gave the asymptotic order on $\omega$ of the Filon method which depends on function values and derivatives at endpoints and stationary points. In the current work, we focus on researching the numerical steepest descent method for the integral (1.1). A new numerical steepest descent method is presented. The asymptotic order on $\omega$ of the new method is consistent with that of the Filon method which also depends on function values and derivatives at endpoints and stationary points. In the following, let us have a brief introduction for the numerical steepest descent method for highly oscillatory integrals.

In 1982, Wong presented a method for computing the Fourier and Bessel transforms [28] by complex integration methods

$$F(x) = \int_0^\infty t^\mu f(t)e^{ixt}dt, \quad \mu > -1, \tag{1.2}$$

$$H_i(x) = \int_0^\infty t^\mu f(t)H_\nu^{(i)}(xt)dt, \quad \mu \pm \nu > -1, \tag{1.3}$$

where $x$ is a real parameter and $H_\nu^{(i)}(t), i = 1, 2,$ are the Hankel functions. Let $\{t_k, w_k\}$ be the nodes and weights of Gauss–Laguerre quadrature rule, if $f(t)$ satisfies some conditions (see [28]), the Fourier transform $F(x)$ can be approximated by

$$F(x) = \frac{x^{\mu+1}}{e^{(\mu+1)i\pi/2}}\sum_{k=1}^{n} w_k f(it_k/x) + \varepsilon_n(f;x), \tag{1.4}$$

where

$$\varepsilon_n(f;x) = \frac{n!\Gamma(n+\mu+1)}{(2n)!x^{2n+\mu+1}}e^{(\mu+1)i\pi/2}f^{(2n)}(i\xi/x), \quad 0 < \xi < \infty. \tag{1.5}$$

The equality (1.5) provides effective asymptotic order

$$\varepsilon_n(f;x) = O\left(\frac{1}{x^{2n+\mu+1}}\right), \quad x \to \infty. \tag{1.6}$$

For the more general $f(t)$, Milovanović [20] considered complex integration methods for evaluating oscillatory integrals on a finite interval

$$\int_a^b f(x)e^{i\omega x}dt. \tag{1.7}$$

An important paper on complex integration methods for oscillatory integrals appeared in 2008 by Huybrechs and Vandewalle [11]. They referred to this method as the numerical steepest descent method. In their work, the researching is extended to the general case

$$\int_a^b f(x)e^{i\omega g(x)}dt. \tag{1.8}$$

The order on $\omega$ of the numerical steepest descent method is given in [11] which is only related to the number of nodes of the suitable Gauss quadrature and the order of the stationary point of the phase function $g(x)$. Based on the work of [11], more deeply comprehensive results on the numerical steepest descent method can be found from the papers [1,3,10].

From the references [14,15,29], it is well known that the error order on $\omega$ in Filon-type method and Asymptotic method can be improved by using the derivative information of $f(x)$ at endpoints and stationary points. In Levin-type method [22], the the error order on $\omega$ also can be improved by using the derivative information of $f(x)$ at endpoints when there is no stationary point. For the case of stationary point, Levin-type method is invalid to approximate $I[f]$. Therefore, adding the derivative information of $f(x)$ at endpoints and stationary points can improve the error order in most of efficient methods for $I[f]$. However, so far, the error order of the existing numerical steepest descent method is independent of function values and derivatives at endpoints and stationary points. The purpose of this paper is to show that the order of the numerical steepest descent method can have similar behaviors with the order of the Filon-type method. Both two orders are related to the function values and derivatives at endpoints and stationary points.

This paper is organized as follows: In Section 2, we will give the numerical scheme of the new method detailedly. Moreover, we will research the convergence order about $\omega$ of the new method. Then, we show the effectiveness of the presented method by some numerical examples in Section 3. Finally, a conclusion is presented in Section 4.

## 2. Descriptions of the new method

In the first of place, we consider the integral (1.1) for the case of no stationary points. Without loss of generality, we let $g(x) = x$ in this case. From [11] and [20], we know the following result.