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A new relaxed PSS preconditioner for nonsymmetric saddle point problems☆

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ABSTRACT

A new relaxed PSS-like iteration scheme for the nonsymmetric saddle point problem is proposed. As a stationary iterative method, the new variant is proved to converge unconditionally. When used for preconditioning, the preconditioner differs from the coefficient matrix only in the upper-right components. The theoretical analysis shows that the preconditioned matrix has a well-clustered eigenvalues around (1, 0) with a reasonable choice of the relaxation parameter. This sound property is desirable in that the related Krylov subspace method can converge much faster, which is validated by numerical examples.

1. Introduction

The steady-state Navier–Stokes system is a basic tool for the modeling of an incompressible Newtonian fluid [30]. Finding effective methods for this system remains a crucial issue for a range of engineering applications [25,26]. Let $\Omega \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) be a bounded, connected domain with a boundary Γ. Given a force field $\mathbf{f}$ and boundary data $\mathbf{g}$, the problem is to ascertain a velocity field $\mathbf{u}$ and a pressure field $\mathbf{p}$ that satisfy the steady-state Navier–Stokes system [15]

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\mathbf{p} = \mathbf{f}, &\quad \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, &\quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity, Δ is the vector Laplacian, ∇ is the gradient and div is the divergence. The Navier–Stokes Eq. (1) is nonlinear due to the existence of $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Two common strategies, namely Picard iteration and Newton iteration, are available for linearizing (1), which leads to the following Oseen problem

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\omega\cdot\nabla)\mathbf{u} + \nabla\mathbf{p} = \mathbf{f}, &\quad \text{in } \Omega, \\ \text{div } \mathbf{u} = 0, &\quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, &\quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

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where ω is obtained from the previous iteration step. Furthermore, the Eq. (2) can be discretized by, say the finite element and finite difference methods, to yield the saddle-point system

$$\begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \quad \text{or} \quad \mathcal{A}u = b, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ corresponds to the discretization of the convection-diffusion term, the full-rank $B \in \mathbb{R}^{m\times n}$ denotes the divergence operator, B^T is the transpose of B (a.k.a. the discrete gradient operator), x, y, f and g are vectors of conforming size. More discretization details can be found in, e.g., [25]. Throughout this paper, it is assumed that A is a nonsingular, nonsymmetric matrix with a positive definite symmetric part and $m \le n$. In what follows, we are concerned with the numerical solution of (3). In the past few years, extensive research has been made in developing effective methods for solving (3). In particular, the iterative methods are favored since the coefficient matrix $\mathcal{A}$ is generally large and sparse. These methods include, e.g., the Uzawa-type variants [2,3,12,13], the HSS-based methods [5,7,9,10,14] and the Krylov subspace methods [6,31]; see [15,25,26] for more detail. Among these candidates, Krylov subspace methods have received much attention for their effectiveness. However, it is known that Krylov subspace methods sometimes suffer from slow convergence or even stagnation. In that case, an appropriate preconditioner is needed to overcome the difficulty. To this end, many preconditioners have been proposed to speed up the convergence of Krylov subspace methods, which include the HSS-type preconditioners [5,7,10,14,21], matrix splitting preconditioners [19,22,29,34], block diagonal (triangular) preconditioners [4,8,20], constraint preconditioners [11,23,24,27] and dimensional splitting preconditioners [16–18]; see [15] for a comprehensive survey.

In this work, we focus on devising a preconditioner for solving (3) which is based on the philosophy of matrix splitting. For nonsymmetric linear systems, the rationale behind developing an effective preconditioner is that it should be preferably as close as possible to the coefficient matrix such that the preconditioned matrix will have a clustered spectrum (away from 0) [15, p.60]. In the context of saddle point system, some efforts have been made towards this goal recently; for instance, Pan et al. [29] present a novel deteriorated form of positive-definite and skew-symmetric splitting (PSS) preconditioner such that the eigenvalues of the preconditioned matrix gather into two clusters, i.e., (0,0) and (2,0), as the relaxation parameter approaches 0. Following the reasoning of [29], Cao et al. [19] propose a relaxed deteriorated PSS preconditioner to ensure that the eigenvalues of the corresponding preconditioned matrix cluster around (1, 0) with the optimal relaxation parameter. As remarked in [19, p.51], however, it is difficult to obtain the optimal parameter during the actual implementation. In practice, a compromise is often made by using other choices of the relaxation parameter; see [19, Section 4] for in-depth discussion. Furthermore, Zhang et al. [34] give a PSS-like iteration scheme that converges unconditionally. It is also proved therein that the preconditioned matrix has eigenvalue 1 with algebraic multiplicity at least n. Nonetheless, how to provide a simple way (in determining the relaxation parameter) such that the eigenvalues related to the preconditioned matrix will cluster around (1, 0) remains a problem. With this background in place, we come up with a new iterative framework pertinent to the saddle point problem (3). When used as an iterative method, the new method is shown to converge unconditionally. If adopted as a preconditioner, it resembles much of $\mathcal{A}$ except for its upper-right entries. Moreover, the preconditioned matrix has a well-clustered eigenvalue distribution (around (1, 0)) which can dramatically accelerate the convergence of the associated Krylov subspace method for (3).

The remainder of this work is organized as follows. In Section 2, we first give a sketch of the deteriorated PSS iteration, then propose the new method and finally verify its unconditional convergence property. In Section 3, we investigate the eigen-info of the preconditioned matrix and the impact upon the convergence of the corresponding Krylov subspace method. In Section 4, we carry out some numerical experiments to validate the effectiveness of the new preconditioner. In Section 5, some conclusions are given. Unless otherwise stated, (block) matrices and preconditioners are denoted by the (calligraphic) uppercase letters while vectors are represented by lowercase ones. Scalars are usually denoted by Greek letters except those, say m, n, i, j and k, which are used for indexing purpose. The imaginary unit are represented by $\mathbf{i}$.

2. A new iteration scheme

The new preconditioner developed in this section hinges on the idea of matrix splitting. Since our work is partly motivated by Pan et al. [29], we shall first give a brief introduction of the deteriorated positive-definite and skew-symmetric splitting (PSS) iteration scheme, and then analyze the convergence property of the proposed method.

2.1. The deteriorated PSS iteration/preconditioner

Based on the idea of matrix splitting, Pan et al. [29] propose the deteriorated PSS iteration scheme for (3). To better introduce our new method in Section 2.2, we sketch out its format in this subsection. More details can be found in [8,29].

Suppose that $\mathcal{A}$ has the following splitting

$$\mathcal{A} = \mathcal{P} + \mathcal{S},$$

where

$$\mathcal{P} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix}.$$