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Gradient superconvergence for a class of semi-cardinal interpolation schemes with cubic and quintic B-splines☆

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ABSTRACT

We solve the conjecture, formulated by Bejancu, Johnson, and Said (2014), on the gradient superconvergence of semi-cardinal interpolation with quintic B-splines, for a hierarchy of finite difference end conditions. We also establish the similar result for cubic B-splines.

1. Introduction

In the recent work [1], Bejancu et al. studied the approximation properties of quintic splines using Schoenberg's 'semi-cardinal interpolation' (SCI) model [2] on the semi-axis domain $[0, \infty)$. For a parameter $h > 0$ and a sequence of real data values $\{y_j\}_0^\infty$, the SCI model requires the spline function s to satisfy the interpolation conditions

$$s(hj) = y_j, \quad j \in \mathbb{Z}_+, \tag{1}$$

on the scaled interpolation grid $h\mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of non-negative integers. Further, the semi-cardinal quintic spline s with knot-set $h\mathbb{Z}_+$ is expressed in [1] via a restricted B-spline series

$$s(x) = \sum_{k=-2}^{\infty} a_k M(h^{-1}x - k), \quad x \geq 0. \tag{2}$$

Here, $M := M_6$ is the quintic (6th order) centered B-spline of class C^4 and support $[-3, 3]$, with knot-set $\mathbb{Z} \cap [-3, 3]$, while the B-spline coefficients $\{a_k\}_{-2}^\infty$ are to be determined from the interpolation conditions (1), two left-end conditions, and growth restrictions at ∞.

Extending the work [3] on cubic splines, the paper [1] formulated an infinite hierarchy of end conditions in terms of B-spline coefficients. Specifically, for a fixed integer $\tau \in \mathbb{Z}_+$, the semi-cardinal quintic spline (2) is required to satisfy two homogeneous finite difference (FD) end conditions of order τ:

$$\Delta^\tau a_{-2} = \Delta^\tau a_{-1} = 0, \tag{3}$$

where Δ is the forward difference operator $\Delta a_k := a_{k+1} - a_k$, with $\Delta^p := \Delta(\Delta^{p-1})$, for $p \geq 1$, and $\Delta^0 a_k := a_k$. As observed in [1], since $s^{(4)}(hk) = h^{-4}\Delta^4 a_{k-2}$, the conditions (3) of orders $\tau = 6,\ 7,\ 8$, and 9 are equivalent to certain end conditions introduced by Behforooz and Papamichael [4, Table (3.20)] in terms of finite differences of 4th order derivatives at knots. In particular, imposing (3) with $\tau = 6$ implies that each of the first two interior knots, at h and $2h$, becomes 'not-a-knot', i.e., $s^{(5)}$ has no discontinuity at these two points.

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For each τ, a unique semi-cardinal quintic spline s satisfying (1)–(3) is constructed in [1], under a polynomial growth restriction for both s and the data sequence $\{y_j\}_0^\infty$. Moreover, let $\kappa := \min\{\tau, 6\}$ and $f \in C^\kappa[0, \infty)$, such that $f^{(\kappa)}$ is bounded on $[0, \infty)$. Then, if s interpolates the values of f on $h\mathbb{Z}_+$, i.e., $y_j := f(hj)$, $j \in \mathbb{Z}_+$, the following uniform approximation rate is proved in [1]:

$$\sup_{x\in[0,\infty)} |f^{(l)}(x) - s^{(l)}(x)| = O(h^{\kappa-l}), \quad \text{as } h \to 0,$$

for $0 \le l \le \kappa - 1$. Hence, if $\tau \ge 6$, the maximal convergence order $O(h^6)$ occurs for $l = 0$ and the uniform error of the first derivative is $O(h^5)$, which match the orders obtained by [4] in the setting of interpolation on a finite uniform grid.

A remarkable additional result of Behforooz and Papamichael [4] is that, for the end conditions corresponding to $\tau = 7$, 8, or 9 in (3), the quintic spline gradient errors at all knots within the domain exhibit the 'superconvergence' order $O(h^6)$, for a sufficiently smooth data function f. The numerical results in quad precision presented in [1] not only confirmed the gradient superconvergence effect for $\tau = 7,\ 8,\ 9$, but also verified this property for $\tau = 10$, although the end conditions generated by $\tau = 10$ do not belong to the class treated in [4]. Based on this evidence, the conjecture was made in [1] that, for semi-cardinal quintic spline interpolation with the FD end conditions (3), the gradient superconvergence order $O(h^6)$ holds at all knots, for any $\tau \ge 7$.

In Section 2, we prove that the statement of this conjecture is true for any data function $f \in C^7[0, \infty)$ with the property that $f^{(7)}$ is bounded on $[0, \infty)$. Further, in Section 3, we establish the corresponding superconvergence result of order $O(h^4)$ for semi-cardinal interpolation with cubic splines.

It is well-known that the property of gradient superconvergence at knots has turned spline interpolation into a popular method for numerical differentiation, with important applications such as the solution of differential equations via collocation. The first proof of this property in the spline literature dates back to Birkhoff and de Boor [5], for cubic spline interpolation with clamped-gradient (or 'complete') end conditions. Kershaw [6] proved that the property also holds for periodic cubic splines, while for other, inferior, end conditions he obtained 'local' superconvergence, in intervals bounded away from the end points. A comprehensive study of local and global superconvergence properties for derivatives of any order and a variety of end conditions in cubic spline interpolation was achieved by Lucas [7]. Recently, Fuselier and Wright [8] showed that gradient superconvergence also holds for a large class of periodic radial basis functions. Our main contribution, in this context, is to demonstrate the utility of semi-cardinal interpolation as a model for the analysis of spline end conditions that exhibit global gradient superconvergence at knots.

2. The quintic spline case

For a fixed $\tau \in \mathbb{Z}_+$, let $\mathcal{S}^+_{5,\tau}$ denote the space of all semi-cardinal quintic splines $s : \mathbb{R}_+ \to \mathbb{R}$ of the form (2) with $h = 1$, that satisfy the end conditions (3). Our proof of gradient superconvergence for semi-cardinal quintic splines rests on the results of [1], which we now summarize.

Specifically, for each $j \in \mathbb{Z}_+$, [1, Theorem 3] established the existence of a unique sequence $\{\mu_{j,k}(\tau)\}_{k\ge-2}$ of real coefficients, such that the function

$$L_{\tau,j}(x) := \sum_{k=-2}^{\infty} \mu_{j,k}(\tau) M(x-k), \quad x \in \mathbb{R}_+, \tag{4}$$

belongs to $\mathcal{S}^+_{5,\tau}$, is exponentially decaying away from j, and satisfies the Lagrange interpolation conditions

$$L_{\tau,j}(j) = 1 \quad \text{and} \quad L_{\tau,j}(k) = 0 \text{ for } k \in \mathbb{Z}_+\backslash\{j\}. \tag{5}$$

Consequently, by [1, Theorem 1], for a data sequence $\{y_j\}_0^\infty$ of polynomial growth at ∞, there exists a unique semi-cardinal quintic spline s of the form (2)–(3), satisfying the interpolation conditions (1) on the scaled grid $h\mathbb{Z}_+$ and obeying the same growth rate. It is expressed as the convergent Lagrange series

$$s(x) = \sum_{j=0}^{\infty} y_j L_{\tau,j}(h^{-1}x), \quad x \in \mathbb{R}_+, \tag{6}$$

and its B-spline coefficients from (2) are given by

$$a_k = \sum_{j=0}^{\infty} y_j \mu_{j,k}(\tau), \quad k \ge -2. \tag{7}$$

The convergence of the last series is ensured by the property [1, Eq. (29)]:

$$|\mu_{j,k}(\tau)| \le K(\tau)\lambda^{|k-j|}, \quad \forall k \ge -2,\ \forall j \in \mathbb{Z}_+, \tag{8}$$

for constants $K(\tau) > 0$ and $\lambda = \frac{1}{2}(z - \sqrt{z^2-4}) \approx 0.43$, where $z = 13 - \sqrt{105}$.