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Note on extremal graphs with given matching number

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ABSTRACT

In this note we explore the extremal graphs with given matching number with respect to topological indices. We present generalization of previous studies on such graphs and apply our findings to various indices that have not yet been considered for graphs with given matching number.

1. Introduction

The so called topological indices are popular descriptors of structural information that have been vigorously studied due to their applications in many fields such as Biochemistry, Information Technology, Phylogeny, and Neural networks [2,9,19].

The problem of finding extremal structures within a certain category of graphs that maximize or minimize a certain index has received much attention [1,2,5,10,11,13,20,21,24,25]. In this note, we focus our attention on graphs with a given matching number.

In this paper, we only consider simple graphs, i.e., a graph without loops or multiedges. For a simple graph G with vertex set V and edge set E, we usually denote by $d_G(u, v)$ (or simply $d(u, v)$) the distance between two vertices u and v of G, and by $d_G(u)$ (or simply $d(u)$) the degree of a vertex u in G. For two graphs G and H, $G \cup H$ denotes the vertex-disjoint union of G and H. $G + H$ denotes the graph obtained from $G \cup H$ by adding all edges between every vertex of G and every vertex of H.

A *matching* in a simple graph G is a set of edges without common vertices. A *maximum matching* is a matching that contains the largest possible number of edges. The *matching number* of a simple graph G, denoted by $\beta(G)$ (or simply β), is the size of a maximum matching of G. The extremal graphs with a given matching number has been studied for spectral radius [8], Wiener index [3,16], Kirchhoff index [25], Zagreb, Harary and hyper-Wiener indices [5], Harary index [6], spanning trees [7], reduced reciprocal Randić [18], Laplacian-energy-like index [23], and many other.

In this note, we investigate the generalization of previous studies on distance-based indices, degree-based indices and apply our findings to other indices.

Let a component of a graph be called *odd* (*even*) if it has odd (even) number of vertices and denote the number of odd components by $o(G)$. The following result named Tutte–Berge formula from [12] is a crucial lemma used in many of such studies and in this note.

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Lemma 1.1. [12] *Let G be a connected graph of order n. Then*

$$n-2\beta = max\{o(G-X)-|X| : X\subset V\}.$$

If $F(G)$ is an index defined on a graph G, we first introduce a general lemma. The proof is very similar to that in [5] and we skip some details.

Lemma 1.2. *For a graph G that is not complete, if $F(G) > F(G+e)$ $(F(G) < F(G+e))$ for any edge $e\notin E$, the minimum (maximum) F(G) among all connected graphs of order n and matching number β is achieved by a graph of the form*

$$\widehat{G} = K_s + (K_{n_1}\cup K_{n_2}\cup\cdots\cup K_{n_t})$$

for some s and t with $s+n_1+\cdots+n_t = n$.

Proof. We only consider the case of $F(G) > F(G+e)$. Let $\widehat{G}$ be a connected graph having minimum $F(G)$ among all connected graphs with n vertices and matching number β. By Lemma 1.1, there exists an $\widehat{X}\subset V(\widehat{G})$ such that

$$n-2\beta = max\{o(\widehat{G}-X)-|X| : X\subset V(\widehat{G})\} = o(\widehat{G}-\widehat{X})-|\widehat{X}|$$

For simplicity, let $|\widehat{X}| = s$ and $o(\widehat{G}-\widehat{X}) = t$. Then $n-2\beta = t-s$.

First suppose that $s=0$, then

$$t = n-2\beta = o(\widehat{G}-\widehat{X}) = o(\widehat{G}) \le 1.$$

When $t = 0$ or 1, $\beta = \lfloor \frac{n}{2}\rfloor$ in this case and the minimum $F(G)$ is achieved by the complete graph.

Now let $s\ge 1$ and hence $t\ge 1$. We claim that there is no even component in $\widehat{G}-\widehat{X}$. Otherwise, by adding an edge in $\widehat{G}$ between a vertex of an even component and a vertex of an odd component of $\widehat{G}-\widehat{X}$, we obtain $\widehat{G}'$ with

$$n-2\beta(\widehat{G}') \ge o(\widehat{G}'-\widehat{X})-|\widehat{X}| = o(\widehat{G}-\widehat{X})-|\widehat{X}| = n-2\beta \ge n-2\beta(\widehat{G}').$$

Thus $\beta(\widehat{G}') = \beta$ and by the condition on $F(\cdot)$, we have $F(\widehat{G}') < F(\widehat{G})$, a contradiction.

Now $\widehat{G}-\widehat{X}$ contains only odd components, denoted by $G_1,\ldots,G_t$. Since the addition of edges will decrease $F(G)$, we can assume that $G_1, G_2,\ldots,G_t$, and the subgraph induced by $\widehat{X}$ are all complete and each vertex of $G_1, G_2,\ldots,G_t$ is adjacent to every vertex in $\widehat{X}$. Let $n_i = |V(G_i)|$ for $i = 1, 2,\ldots,t$. The conclusion follows. □

Remark. Also as an immediate consequence of the condition of Lemma 1.2, it is easy to see that

$$F(K_n)\le F(G)\le F(T)$$

if $F(G) > F(G+e)$; and

$$F(K_n)\ge F(G)\ge F(T)$$

if $F(G) < F(G+e)$, for any spanning tree T of any connected graph G on n vertices.

In the rest of this note we make use of Lemma 1.2 to characterize or partially characterize extremal graphs with generally defined distance-based indices, degree-based indices, and some other indices not yet studied for graphs with given matching number.

2. Distance-based indices

The most well known distance-based index is probably the Wiener index, defined as

$$W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$$

and introduced by Wiener [22]. In recent years many variations of the Wiener index have been introduced and found to be of interest because of their correlations with physical properties of different chemical compounds. Let

$$F(G) = \frac{1}{2}\sum_{u,v\in V(G)} f(d(u,v))$$

for some strictly increasing (decreasing) function f. It is easy to see that Lemma 1.1 holds for F as the addition of an edge will only decrease (increase) the distances between vertices. The extremal structure can be further characterized.

Theorem 2.1. *Among connected graphs of order n and matching number β,*

$$F(G) = \frac{1}{2}\sum_{u,v\in V(G)} f(d(u,v))$$

is minimized (maximized) by

$$\widehat{G} = K_s + (K_{2\beta-2s+1}\cup \overline{K_{n+s-2\beta-1}})$$