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Weak solvability of a fractional viscoelastic frictionless contact problem☆

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ABSTRACT

The goal of this paper is to study a quasistatic frictionless contact problem for a viscoelastic body in which the constitutive equation is modeled with the fractional Kelvin–Voigt law and the contact condition is described by the Clarke subdifferential of a nonconvex and nonsmooth functional. The variational formulation of this problem is provided in the form of a fractional hemivariational inequality. In order to solve this inequality, we apply the Rothe method and prove that the associated abstract Volterra inclusion has at least one solution.

1. Introduction

Fractional calculus, which is a natural generalization of traditional integer-order calculus into arbitrary order, has been of great interest recently. This is due to the intensive development of applications of fractional calculus to many physical and engineering fields such as solid and fluid mechanics, electrochemistry, control systems and dynamical systems, etc., one may refer to [2,7,15,16,19,20,27,31]. In the field of material science, in the last few years, fractional calculus has been frequently applied to describe the constitutive laws for viscoelastic materials, see, e.g., [7,15,20]. The reason is that, compared with the traditional integer-order derivative, the definition of the fractional derivative includes the integration of a function, i.e., the fractional derivative of the function at some time depends not only on the current time, but also on the history of the process. This fact is consistent with the rheological behavior of viscoelastic materials, cf. [8]. For the specific viscoelastic materials which meet the fractional constitutive models, we refer to [10,25,28,30].

Contact mechanics has many applications in industry and our daily lives, such as locomotive wheel-rail contact, braking systems, bearings, gasket seals, combustion engines, mechanical linkages, ultrasonic welding, metalworking, and so on. Because of the attractiveness of these processes, modeling, mathematical analysis and numerical simulations of various contact processes become particularly popular. As a result, a general mathematical theory of contact mechanics is currently emerging. It provides a sound, clear and rigorous mathematical background to the corresponding contact model, proving existence, uniqueness, regularity results, etc. Many mathematical tools are employed in the theory, such as nonlinear

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inclusions, variational inequalities and hemivariational inequalities, etc. We also would like to mention that the notion of hemivariational inequality was introduced by Panagiotopoulos in the 1980s as a generalization of variational inequality. It plays an important role in describing many mechanical problems arising in solid mechanics. For motivations, mathematical results on hemivariational inequalities and their applications to mechanics, we refer to [11–13,22–24,26] and the references therein.

Although the theory of fractional viscoelasticity and contact mechanics has been considerable developed in the past few years, and delivered numerous papers and monographs, as far as we know, there are no papers and monographs focused on contact problems for the viscoelastic solid materials with fractional viscoelastic constitutive laws. Motivated by the above, the aim of this paper is to study the existence of the weak solutions for a viscoelastic frictionless contact problem with the fractional Kelvin–Voigt constitutive law. The novelty of this work consists in the fact that we consider the viscoelastic materials with the fractional Kelvin–Voigt constitutive law and the contact boundary is modeled with the Clarke subdifferential of a nonconvex and nonsmooth function. In a consequence, our new contact model leads to the study of a new class of fractional hemivariational inequalities, which are associated with an abstract Volterra inclusion.

The paper is structured as follows. We use the Rothe method to study the existence of solutions for an abstract Volterra inclusion with the Clarke subdifferential in Section 2. In Section 3, in order to facilitate the reader, we provide a detailed analysis of the rheological behavior for the fractional Maxwell model and the fractional Kelvin–Voigt model, respectively. In Section 4, we present a new mathematical model with the fractional Kelvin–Voigt constitutive law and derive its variational formulation which has the form of a fractional hemivariational inequality for the displacement field. The weak solvability of this hemivariational inequality is obtained by using the abstract results of Section 2. Finally, in Appendix we recall some necessary definitions and results which are useful in the proof of the main result.

2. Solvability of an abstract Volterra inclusion with the Clarke subdifferential

Let V and X be two separable and reflexive Banach spaces, and $(0, T)$ be a finite time interval of interest. We denote by M a linear, bounded, and compact operator from V into X, and by $M^*: X^* \to V^*$ its adjoint.

Given $u_0 \in V$ and $k_i \in L^\infty(0, T)$, $k_i(t) > 0$ for a.e. $t \in (0, T)$, $i = 1, 2$, we consider the convolution (Volterra) operator K_i: $L^2(0, T; V) \to C(0, T; V)$ defined by

$$(K_i v)(t) = u_0 + \int_0^t k_i(t-s)v(s)\,ds \quad \text{for} \quad v \in L^2(0,T;V),$$

for all $t \in (0, T)$, $i = 1, 2$.

The main problem under consideration in this section is formulated as follows.

Problem 2.1. Find $v \in L^2(0,T;V)$ such that

$$Av(t) + B\big((K_1v)(t)\big) + M^*\partial J\big(M(K_2v)(t)\big) \ni f(t) \quad \text{for a.e.} \quad t \in (0,T), \tag{1}$$

where A, B: $V \to V^*$ are single-valued operators, $J : X \to \mathbb{R}$ is a functional, ∂J stands for its Clarke subdifferential and $f \in L^2(0, T; V^*)$.

We recall that a function $v \in L^2(0,T;V)$ is called a solution to Problem 2.1 if and only if there exists $\zeta \in L^2(0, T; X^*)$ such that

$$Av(t) + B\big((K_1v)(t)\big) + M^*\zeta(t) = f(t) \quad \text{for a.e.} \quad t \in (0,T)$$

with

$$\zeta(t) \in \partial J\big(M(K_2v)(t)\big) \quad \text{for a.e.} \quad t \in (0,T).$$

Remark 2.2. Let us introduce the operators $\mathcal{A}$, $\mathcal{B} : L^2(0,T;V) \to L^2(0,T;V^*)$ and $\mathcal{M} : L^2(0,T;V) \to L^2(0,T;X)$ defined by

$$(\mathcal{A}v)(t) = Av(t), \quad (\mathcal{B}v)(t) = B\big((K_1v)(t)\big), \quad (\mathcal{M}v)(t) = M(v(t)) \tag{2}$$

for $v \in L^2(0,T;V)$ and a.e. $t \in (0, T)$. The operators $\mathcal{A}$ and $\mathcal{M}$ are the Nemytski (superposition) operators corresponding to A and M, respectively. With these notation, it can be observed that the function $v \in L^2(0,T;V)$ is a solution of Problem 2.1 if and only if there exists $\zeta \in L^2(0, T; X^*)$ such that

$$\mathcal{A}v + \mathcal{B}v + \mathcal{M}^*\zeta = f$$

with $\zeta(t) \in \partial J\big(M(K_2v)(t)\big)$ for a.e. $t \in (0, T)$.

In order to derive a result on existence of solution to Problem 2.1, we introduce the hypotheses on the operators A, B: $V \to V^*$ and the functional $J : X \to \mathbb{R}$.

<u>$H(A)$</u> : The operator A: $V \to V^*$ satisfies

(a) $A \in \mathcal{L}(V, V^*)$ and monotone,