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Necessary conditions for the convergence of subdivision schemes with finite masks

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ABSTRACT

Knowing that the convergence of the subdivision scheme with a nonnegative mask relies on the location of its support of the mask, we consider the positions of the points in the support and the convex cover of the support. We demonstrate the different properties between the inner and boundary points of the support for the mask, when the corresponding subdivision scheme converges. Furthermore, we find out that the so-called connectivity of a matrix A deduced by a given mask is some simple condition to guarantee those properties for nonnegative masks.

1. Introduction

Subdivision schemes are widely used in surface modeling in computer aided geometric design (CAGD) and the animation industry. These schemes are also intimately connected to wavelet bases and their associated fast bank algorithms [5]. Moreover, these schemes can be used in recursive refinements of given control points whose limit turns to be a desired visually smooth object.

Denote $\mathbb{Z}^s$ to be the integer lattice. A subdivision scheme is defined by a fixed finitely supported real sequence (mask) $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$. We should denote the support of the mask by $\Omega = \{\alpha : a(\alpha) \neq 0\}$ and $[\Omega]$ the convex cover of Ω. The boundary of the convex cover $[\Omega]$ formed by Ω will be denoted by $\partial[\Omega]$. Thus, $[\Omega]$ is a polytope. An $(s-1)$-dimensional face S_{s-1} is a facet of $[\Omega]$ and for $0 \le j < s-1$, a j-dimensional face S_j is a facet of a $(j+1)$−dimensional face S_{j+1} of $[\Omega]$. Moreover, denote for a given mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ the set

$$\mathcal{A}(\lambda) = \{\alpha : a(\alpha) \neq 0 \text{ and } \alpha \equiv \lambda \pmod 2\}, \quad \forall \lambda \in \mathbb{Z}^s,$$

and $|\mathcal{A}(\lambda)|$ to be the number of the elements in the set $\mathcal{A}(\lambda)$.

Given an initial finite sequence of data values, $v^0 = \{v^0(\alpha)\}$, a subdivision scheme with a mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ defines a sequence of values $v^k(\alpha)$ recursively by the rule

$$v^k(\alpha) = \sum_\beta v^{k-1}(\beta) a(\alpha - 2\beta).$$

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This scheme is said to be convergent if for each v^0 there exists a continuous function f such that

$$\lim_{k\to\infty} \sup_{\alpha} |f\left(\frac{\alpha}{2^k}\right) - v^k(\alpha)| = 0 \tag{1.1}$$

and $f \not\equiv 0$ for at least one v^0.

Let H be the hat function defined by $H(y) = 1 - |y|$, if $|y| \le 1$ and 0 for all other y. For $x = (x_1, \ldots, x_s)^T \in \mathbb{R}^s$ we set $\psi(x) = H(x_1)\cdots H(x_s)$. Using $v^k(\alpha)$ we get a "polygon" $f^k(x) = \sum_\beta v^k(\beta)\psi(2^k x - \beta)$. Clearly, $f^k(\beta/2^k) = v^k(\beta)$ and therefore the convergence of the subdivision scheme is equivalent to the uniform convergence of f^k. On the other hand, write $a^1(\alpha) = a(\alpha)$ and $a^k(\alpha) = \sum_\beta a^{k-1}(\beta)a(\alpha - 2\beta)$. In particular, taking $v^0(\alpha) = \delta_0(\alpha)$, where $\delta_0(0) = 1$ and $\delta_0(\alpha) = 0$ if $\alpha \ne 0$, one has $f^k(x) = \sum_\beta a^k(\beta)\psi(2^k x - \beta)$. Therefore, the convergence of the subdivision scheme is equivalent to the uniform convergence of

$$\sum_\beta a^k(\beta)\psi(2^k x - \beta). \tag{1.2}$$

In this paper, when we say the subdivision scheme converges to φ, we mean (1.2) converges to φ, which is also equivalent to the scheme with δ_0 converges to φ. A comprehensive discussion of this subject can be found in [1].

The convergence of the subdivision schemes with finite mask has been characterized in [3,4,10]. The result can be summarized as follows.

Theorem 1.1. *A subdivision scheme associated with a fixed finitely supported real sequence (mask) $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ converges if and only if*

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha + 2\beta) = 1, \quad \forall\, \alpha \in \mathbb{Z}^s \tag{1.3}$$

and

$$\lim_{k\to\infty} \sup_{\alpha\in\mathbb{Z}^s,\, e\in E^s} |a^k(\alpha) - a^k(\alpha - e)| = 0, \tag{1.4}$$

where $e \in E^s :=$ extreme points of $[0, 1]^s$, i.e., $E^s = \{(\delta_1, \ldots, \delta_s)^T : \delta_i \in \{0, 1\}, i = 1, \ldots, s\}$.

The first condition (1.3) is called the sum rule in the literature and it is clear and easy to check. However, the second one is rather difficult to verify. It can be described by means of the so-called joint spectral radius of some square matrices (see [1,2,4]). However, by a result in [9] the decision problem of joint spectral radii is generally NP-hard.

In 2005 the uniform convergence of nonnegative univariate subdivision has been completely characterized in [11]. The result is as follows.

Theorem 1.2. *Let $\{a(j) : j = 0, \ldots, N\}$ be a nonnegative mask, which satisfies $a(0), a(N) \ne 0$. Then the univariate subdivision scheme associated with this mask converges if and only if*

(1) *$\sum_j a(2j) = \sum_j a(2j + 1) = 1$ and $0 < a(0)$, $a(N) < 1$, and*
(2) *the greatest common divisor of $\{j : a(j) \ne 0\}$ is 1, i.e. $\gcd\{j : a(j) \ne 0\} = 1$.*

Thus, if the univariate subdivision scheme with the nonnegative mask converges, then the boundary points of the corresponding support satisfy $0 < a(0) < 1$ and $0 < a(N) < 1$. Furthermore, $|a(0)| < 1$ and $|a(N)| < 1$ without the restriction of the positivity (see (1.4)). Thus, it is of some interest to consider whether the boundary points of the support have the similar behavior for the multivariate case. Indeed we get more involved new results.

Theorem 1.3. *Let $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ be a finite mask in $\mathbb{R}^s$. Assume that the subdivision scheme associated with $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ converges to a continuous function φ. If $|\mathcal{A}(\lambda)| = 1$ for some $\lambda \in \mathbb{Z}^s$, then the only one element of $\mathcal{A}(\lambda)$ (say α') belongs to $\Omega\backslash\partial[\Omega]$ and $\varphi(\alpha') = 1$. Furthermore, for any j-dimensional face S_j of the polytope $[\Omega]$ and $0 \le j < s$ there holds*

$$|\sum_{\alpha\in S_j\cap\Omega} a(\alpha)| < 2^j. \tag{1.5}$$

If, in addition, the mask $\{a(\alpha) : \alpha \in \mathbb{Z}^s\}$ is nonnegative, then there is at most one $\beta \in \mathbb{Z}^s$ satisfying $\varphi(\beta) = 1$ and

$$0 < \sum_{\alpha\in S_j\cap\Omega} a(\alpha) < 2^j, \quad 0 \le j < s. \tag{1.6}$$

However, it is unknown, whether one can use some simple conditions to guarantee these properties. We will demonstrate that for the case of nonnegative masks the so-called connectivity of a matrix A deduced by given mask is the suitable condition. To this end, let A be the square matrix defined by

$$A(\alpha, \beta) = a(-\alpha + 2\beta), \quad \alpha, \beta \in [\Omega] \cap \mathbb{Z}^s. \tag{1.7}$$

It means that $\alpha \in [\Omega]$ and $a(-\alpha + 2\beta) \ne 0$ (i.e., $-\alpha + 2\beta \in \Omega$) implies $\beta = \frac{\alpha + (-\alpha + 2\beta)}{2} \in [\Omega]$. Hence, in case of a convergent scheme, the entries in each row of A sum up to one, and in case of a nonnegative mask, A is row stochastic. Let