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On the boundary controllability of a semilinear degenerate system with the convection term

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ABSTRACT

This paper concerns the approximate controllability of a semilinear degenerate equation with the convection term in one dimensional space. The control acts on the 'degenerate' part of the boundary. By using Kakutani fixed point theorem, the control system is shown to be approximately controllable.

1. Introduction

Controllability theory has been widely investigated for partial differential equations. Particularly, for the classical parabolic equations, we refer to [2,12,13,15] and the references therein. The controllability problems, such as exact controllability, null controllability and approximate controllability, are classical problems in control theory. If the control acts on a nonempty subset of the domain where the solutions are defined, we call it interior control; if the control acts on the boundary of the domain where the solutions are defined, we call it boundary control. Roughly speaking, the exact controllability can be formulated as follows. For any given initial and final states, there exists a control such that the solution approaches the final state at a fixed finite time. If the final state is zero, the problem is null controllable. If the solution approaches the final state at a fixed finite time approximately, the problem is approximately controllable.

Recently, the controllability of degenerate parabolic systems has been studied in the literature [1,3–8,10,11,14,18–22] and the references therein. In 2004, Cannarsa et al. considered the following degenerate parabolic equation

$$u_t - (x^\alpha u_x)_x + q(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T = (0,1)\times(0,T),$$

where $\omega = (a,b)$ is a nonempty subset of $(0,1)$ and $h \in L^2(Q_T)$ is the interior control which acts on ω. The degenerate equation comes from many physical problems, such as Prandtl equations. According to the degeneracy, the problem is divided into the weak degenerate problem and the strong degenerate problem. Precisely, when $0 < \alpha < 1$, the problem is a weak degenerate problem, and the boundary condition is

$$u(0,t) = u(1,t) = 0, \quad t \in (0,T);$$

when $\alpha \geq 1$, the problem is a strong degenerate problem, and the boundary condition is

$$x^\alpha u_x(0,t) = 0, \quad u(1,t) = 0, \quad t \in (0,T).$$

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It is shown that the problem is null controllable if $0 < \alpha < 2$ [1,3–7,18], while not if $\alpha \geq 2$ [8]. Moreover, Cannarsa et al. [8] and Wang [19] proved the regional and the persistent reginal null controllability and the approximate controllability for $\alpha > 0$, respectively. In [14], Flores and Teresa considered the equation with the gradient term

$$u_t - (x^\alpha u_x)_x + x^{\alpha/2} b u_x + cu = h(x,t)\chi_\omega, \quad (x,t) \in Q_T,$$

where $b, c \in L^\infty(Q_T)$. They proved the null controllability for $0 < \alpha < 2$. For the semilinear equation

$$u_t - (x^\alpha u_x)_x + f(x,t,u,u_x) = h(x,t)\chi_\omega \quad (x,t) \in Q_T$$

with $f(x, t, u, \eta)$ satisfying $|f_\eta(x, t, u, \eta)| \leq Cx^{\alpha/2}$ for some constant $C > 0$ and some other suitable conditions Cannarsa and Fragnelli [3] studied the regional and the persistent reginal null controllability for $0 < \alpha < 2$, while Du et al. [11] investigated the approximate controllability for each $\alpha > 0$. For the semilinear equation with the convection term

$$u_t - (x^\alpha u_x)_x + (p(x,t,u))_x + q(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in Q_T,$$

Wang and Du [21,22] proved the null controllability for $0 < \alpha < 1/2$ and the approximate controllability for $\alpha > 0$, respectively.

As we know, for the non-degenerate parabolic equations, boundary controllability can be derived from interior controllability. For the degenerate parabolic equation, if the control acts on 'non-degenerate' part of the boundary, we can also get the controllability from interior controllability by using the method for the non-degenerate problem. Indeed, we extend the coefficients to smooth functions on a new domain $\tilde{\Omega}$, which is bigger than the original domain Ω. Let the control act on $\omega \subset \tilde{\Omega} \backslash \Omega$. Then boundary controllability can be shown from the interior controllability. However, if the control acts on 'degenerate' part of the boundary, we cannot get the controllability by the same method.

In this paper, we investigate the boundary controllability of the semilinear system with convection term

$$u_t - (x^\alpha u_x)_x + (p(x,t,u))_x + q(x,t,u) = 0, \quad (x,t) \in Q_T, \tag{1.1}$$

$$u(0,t) = h(t)\chi_{[T_1,T_2]}, \quad u(1,t) = 0, \quad t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{1.3}$$

where $0 < \alpha < 1$, $u_0 \in L^2(0, 1)$, $h \in L^\infty(0, T)$ is the control function, χ is the characteristic function, $0 < T_1 < T_2 < T$. p, q are measurable functions satisfying the following conditions:

(i) There exists a positive constant M, such that

$$x^{-\alpha/2}|p(x,t,u) - p(x,t,v)| + |q(x,t,u) - q(x,t,v)| \leq M|u-v|, \quad (x,t) \in Q_T,\ u, v \in \mathbb{R};$$

(ii) $x^{-\alpha/2}p(\cdot,\cdot,0), q(\cdot,\cdot,0) \in L^2(Q_T)$;

(iii) $p(x, t, u)$, $q(x, t, u)$ are differentiable at $u = 0$.

The assumptions on p show that the convection term vanishes on 'degenerate' part of the boundary. We prove the approximate controllability of the system (1.1)–(1.3). That is to say, for any final state $u_d \in L^2(0, 1)$ and any admissible error value $\varepsilon > 0$, there exists a control h such that the solution u to the problem (1.1)–(1.3) satisfies

$$\|u(\cdot,T) - u_d(\cdot)\|_{L^2(0,1)} \leq \varepsilon. \tag{1.4}$$

This paper is inspired by Cannarsa et al. [9,10]. In [9], Cannarsa et al. showed approximate controllability of the problem (1.1)–(1.3) with $p(x,t,u) = q(x,t,u) = 0$, while Du [10] investigated the approximate controllability of the problem (1.1)–(1.3) with $p(x,t,u) = 0$. We first study the linear system and construct the control by the adjoint problem, then we study the semilinear system by using the Kakutani fixed point theorem. However, there are many essential difficulties. On the one hand, since the control function is in $L^\infty(0, T)$, the weak solution has poor regularity and we cannot get the L^2 estimates of $x^{\alpha/2}u_x$ and u_t. On the other hand, when using the fixed point theorems to prove the well-posedness and the approximate controllability of the semilinear problem, we need some compactness estimates, which are more precise and complex caused by the convection term.

The paper is organized as follows. In Section 2, we establish the well-posedness of the linear problem. In Section 3, we study the convergence of the weak solutions to the linear problems. Subsequently, we establish the approximate controllability of the linear system and the semilinear system in Sections 4 and 5, respectively.

2. Well-posedness of the linear problem

In this section, we establish the well-posedness of the following linear problem

$$u_t - (x^\alpha u_x)_x + (x^{\alpha/2} b u)_x + cu = f + (x^{\alpha/2} g)_x, \quad (x,t) \in Q_T, \tag{2.1}$$

$$u(0,t) = h(t)\chi_{[T_1,T_2]}, \quad u(1,t) = 0, \quad t \in (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1), \tag{2.3}$$

where $b, c \in L^\infty(Q_T)$, $f, g \in L^2(Q_T)$, $h \in L^2(0, T)$ and $u_0 \in L^2(0, 1)$.