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The dimensional splitting iteration methods for solving saddle point problems arising from time-harmonic eddy current models☆

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ABSTRACT

A dimensional splitting iteration method is proposed for solving the saddle point problems arising from the finite element discretization of the hybrid formulation of the time-harmonic eddy current models, which is by making use of the special positive semidefinite splittings of the saddle point matrix. It is proved that the proposed iteration method is unconditionally convergent for both cases of simple topology and general topology. Numerical results show that the corresponding preconditioner is superior to the existing preconditioners, when those preconditioners are used to accelerate the convergence rate of Krylov subspace methods.

1. Introduction

To simulate the electromagnetic phenomena concerning alternating currents at low frequencies, it often uses the time-harmonic eddy current model. The hybrid formulation of the complete eddy current model proposed and analyzed in [1], can be precisely described as the following system of equations

$$\begin{cases} \mathbf{curl}\ (\sigma^{-1}\mathbf{curl}\ \mathbf{H}_C) + \mathrm{i}\omega\mu\mathbf{H}_C = \mathbf{curl}\ (\sigma^{-1}\mathbf{J}_{e,C}) & \text{in} \quad \Omega_C, \\ \mathbf{curl}\ (\mu^{-1}\mathbf{curl}\ \mathbf{E}_I) = -\mathrm{i}\omega\mathbf{J}_{e,I} & \text{in} \quad \Omega_I, \\ \mathrm{div}(\epsilon\mathbf{E}_I) = 0 & \text{in} \quad \Omega_I, \\ \mu^{-1}\mathbf{curl}\ \mathbf{E}_I \times \mathbf{n} = 0 & \text{on} \quad \partial\Omega, \\ \epsilon\mathbf{E}_I \cdot \mathbf{n} = 0 & \text{on} \quad \partial\Omega, \\ \mathbf{H}_C \times \mathbf{n} = (-\mathrm{i}\omega\mu)^{-1}\mathbf{curl}\ \mathbf{E}_I \times \mathbf{n} & \text{on} \quad \Gamma, \\ \mathbf{E}_I \times \mathbf{n} = \sigma^{-1}(\mathbf{curl}\ \mathbf{H}_C - \mathbf{J}_{e,C}) \times \mathbf{n} & \text{on} \quad \Gamma. \end{cases} \tag{1.1}$$

For the sake of simplicity, the computational domain $\Omega \subset \mathbb{R}^3$ is assumed to be a simply connected Lipschitz polyhedron. Suppose the conducting region Ω_C strictly contained in Ω and its complement $\Omega_I := \Omega \setminus \overline{\Omega_C}$. We shall assume that Ω_C and Ω_I are Lipschitz polyhedrons and that Ω_C is connected but not necessarily simply connected. The magnetic permeability μ is assumed to be a uniformly positive definite 3×3 tensor with entries in $\mathbb{L}^\infty(\Omega)$, whereas the electric conductivity σ is supposed to be a positive definite tensor in the conducting regions, and to be null in non-conducting regions. The dielectric

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permittivity ϵ is also assumed to be a uniformly positive definite symmetric tensor. The real scalar constant $\omega \neq 0$ is a given angular frequency. In addition, the symbol i denotes the imaginary unit, i.e., $\mathrm{i}=\sqrt{-1}$, $\partial\Omega$ denotes the boundary of the domain Ω, $\Gamma := \overline{\Omega}_C \cap \overline{\Omega}_I$, $\mathbf{n}_{|\partial\Omega}$ and $\mathbf{n}_{|\Gamma}$ represent the unit outward normal vectors on Ω and on Γ pointing toward Ω_I, respectively. Without causing confusion, we use $\mathbf{n}$ to simply represent $\mathbf{n}_{|\partial\Omega}$ and $\mathbf{n}_{|\Gamma}$. Here, $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{J}_e$ are the electric field, the magnetic field and a given generator current, respectively. For a given vector field $\mathbf{v}$ defined in Ω, we denote by $\mathbf{v}_L$ the restriction to Ω_L, $L = C, I$.

The hybrid formulation (1.1) uses as main unknowns the magnetic field in the conductor and the electric field in the insulator. It is a convenient approach for complicated geometrical configurations where the conductor Ω_C is not simply connected, as unrelated or nonmatching meshes on Ω_C and Ω_I are possibly allowed to be used in its finite element approximation. See [1–4] for more details.

By using the finite element method to discretize the hybrid formulation (1.1) (see [1,2]), we can obtain the following linear system

$$\begin{pmatrix} S_C+\mathrm{i}M_C & D^{\mathrm{T}} & B_C^{\mathrm{T}} & 0 \\ D & \mathrm{i}S_I & 0 & B_I^{\mathrm{T}} \\ B_C & 0 & 0 & 0 \\ 0 & B_I & 0 & 0 \end{pmatrix} \begin{pmatrix} H_C \\ \widetilde{E}_I \\ Q \\ \Phi_I \end{pmatrix} = \begin{pmatrix} F_C \\ G_I \\ 0 \\ 0 \end{pmatrix}, \tag{1.2}$$

where $S_C \in \mathbb{R}^{n_1\times n_1}$ and $S_I \in \mathbb{R}^{n_2\times n_2}$ are symmetric positive semidefinite matrices, $M_C \in \mathbb{R}^{n_1\times n_1}$ is a symmetric positive definite matrix, $B_C \in \mathbb{R}^{m_1\times n_1}$ and $B_I \in \mathbb{R}^{m_2\times n_2}$ have full row ranks and $D \in \mathbb{R}^{n_2\times n_1}$ is a real matrix. Let $\widetilde{\mathbf{E}}_I$ be a suitable magnetic vector potential such that $\mathbf{H}_I = -(\mathrm{i}\omega\mu)^{-1}\mathbf{curl}\ \widetilde{\mathbf{E}}_I$ and q, ϕ_I be two Lagrange multipliers. The complex valued unknown vectors H_C, $\widetilde{E}_I$, Q and Φ_I are the coefficients of the finite element approximations to $\mathbf{H}_C$, $\widetilde{\mathbf{E}}_I$, q and ϕ_I in the chosen bases of certain finite element subspaces, respectively. The complex valued right-hand side vectors F_C and G_I are obtained by applying the functionals f_C and g_I to the elements of the bases of certain finite element subspaces, with f_C and g_I being defined through proper integrals with respect to $\sigma^{-1}\mathbf{J}_{e,C}$ and $\mathbf{J}_{e,\,I}$, respectively. For more details about these complex valued unknown and right-hand side vectors, we refer to [1,3].

The first Betti number of Ω_I is a topological invariant measuring the number of nonbounding cohomologically independent cycles in Ω_I (see, e.g., [5–7]), and its value plays an important role in the solution of the saddle point problem (1.2). When the first Betti number of Ω_I is equal to zero, the saddle point problem (1.2) is referred to as simple topology; when the first Betti number of Ω_I is greater than zero, the saddle point problem (1.2) is referred to as general topology; see [3].

(1) The case of simple topology

From the point of algebra, the first Betti number of Ω_I is equal to zero if and only if the matrix

$$\mathcal{A}_I = \begin{pmatrix} \mathrm{i}S_I & B_I^{\mathrm{T}} \\ B_I & 0 \end{pmatrix} \tag{1.3}$$

is nonsingular. In addition, since the matrix S_I is symmetric positive semidefinite, we can easily show that $\mathcal{A}_I$ is nonsingular if and only if the following two conditions hold

(i) $\mathrm{null}(S_I) \cap \mathrm{null}(B_I) = \{0\}$;

(ii) $\mathrm{null}(B_I^{\mathrm{T}}) = \{0\}$.

Here, we denote by null($\cdot$) the kernel space of the corresponding matrix. Note that these two conditions readily imply that $S_I + \tau B_I^{\mathrm{T}} B_I$ is symmetric positive definite for any constant $\tau > 0$. When the matrix S_I is symmetric positive definite, we may set $\tau = 0$; otherwise, we may let $\tau > 0$.

By pre-multiplying $-\mathrm{i}$ on both sides of the saddle point problem (1.2), utilizing the equation $B_I\widetilde{E}_I = 0$, and using the zero patterns in the coefficient matrix as well as the right-hand side vector, we can equivalently transform (1.2) into the following linear system

$$\mathcal{A}u := \begin{pmatrix} A_1 & -\mathrm{i}D^{\mathrm{T}} & B_C^{\mathrm{T}} & 0 \\ -\mathrm{i}D & A_2 & 0 & B_I^{\mathrm{T}} \\ -B_C & 0 & 0 & 0 \\ 0 & -B_I & 0 & 0 \end{pmatrix} \begin{pmatrix} H_C \\ \widetilde{E}_I \\ -\mathrm{i}Q \\ -\mathrm{i}\Phi_I \end{pmatrix} = \begin{pmatrix} -\mathrm{i}F_C \\ -\mathrm{i}G_I \\ 0 \\ 0 \end{pmatrix} := b, \tag{1.4}$$

where

$$A_1 = M_C - \mathrm{i}S_C, \quad A_2 = S_I + \tau B_I^{\mathrm{T}} B_I.$$

That is

$$\mathcal{A}u := \begin{pmatrix} A & B^{\mathrm{T}} \\ -B & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} := b, \tag{1.5}$$