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Symmetries, travelling-wave and self-similar solutions of the Burgers hierarchy

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ABSTRACT

We examine the general element of the Burgers Hierarchy, $u_t+\frac{\partial}{\partial x}(\frac{\partial}{\partial x}-u)^n u=0,\ \ n=0,1,2,\dots,$ for its Lie point symmetries. We use these symmetries to construct traveling-wave and self-similar solutions. We observe that the general member of the hierarchy can be rendered as a linear $(1+1)$-evolution equation by means of an elementary Riccati transformation and examine this equation for its Lie point symmetries. With the use of these symmetries we can construct the traveling-wave and self-similar solutions in closed form.

1. Introduction

Burgers equation,

$$u_t+2uu_x+u_{xx}=0, \tag{1}$$

was introduced by Burgers [7] in the late forties to describe wave processes in acoustics and hydrodynamics to turbulent modes and has found many applications. In fact, the equation was known to Forsyth [13] and had been discussed by Bateman [3] a few years later. However, it is normally known as Burgers equation as a consequence of the amount of work on the equation performed by Burgers. Although, (1) can be linearised to the standard Heat Equation by means of a procedure known as the Hopf–Cole transformation [8,15] its direct study has attracted and continues to attract considerable attention [4–6,12,16,24,25,32]. Possibly the main reason for this attention is that there exists an hierarchy of partial differential equations based on (1) [18,30]. The hierarchy is defined according to the formula

$$u_t+\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}-u\right)^n u=0,\quad n=0,1,2,\dots. \tag{2}$$

In this paper, we report the Lie point symmetries of the members of the hierarchy, construct three families of reduced ordinary differential equations, examine their symmetry properties and investigate their integrability in terms of Singularity Analysis. Equation for the Burgers hierarchy has been intensively studied for the last several years. For example, self–similar

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solutions and the Cauchy problem for this equation were studied. Lie symmetries for the second member of this hierarchy were obtained as well [19,20,22,23].

2. Symmetry analysis

For the calculations of the symmetries we use the Mathematica add-on, sym [2,9–11]. Eq. (1) possesses the five Lie point symmet ries

$$\begin{aligned}
\Gamma_1 &= \partial_t \\
\Gamma_2 &= \partial_x \\
\Gamma_3 &= 2t\partial_x + \partial_u \\
\Gamma_4 &= 2t\partial_t + x\partial_x - u\partial_u \\
\Gamma_5 &= 2t^2\partial_t + 2tx\partial_x + (x - 2tu)\partial_u
\end{aligned}$$

which has the Lie algebra $sl(2, R)\oplus_s 2A_1$, *ie* $A_{5,\,40}$ in the Mubarakzyanov and Morozov classification scheme [26–29].

When we turn to the second member of the hierarchy, namely .

$$u_t + u_{xxx} + 3u_x^2 + 3uu_{xx} + 3u^2u_x = 0. \tag{3}$$

This equation was derived for the description of nonlinear waves in several physical systems [17,21]. We find the three symmetries,

$$\{\partial_t, \partial_x, 3t\partial_t + x\partial_x - u\partial_u\},$$

which has the algebra $A_{3,5}^{1/3}$. In the case of the third member of the hierarchy,

$$u_t + u_{xxxx} + 4uu_{xxx} + 10u_xu_{xx} + 6u^2u_{xx} + 12uu_x^2 + 4u^3u_x = 0, \tag{4}$$

we obtain the symmetries

$$\{\partial_t, \partial_x, 4t\partial_t + x\partial_x - u\partial_u\}.$$

In this case the algebra is $A_{3,5}^{1/4}$.

These results prompt the following theorem.

Theorem 2.1. *The symmetries of the nth member of the Burgers hierarchy are*

$$\{\partial_t, \partial_x, (n+1)t\partial_t + x\partial_x - u\partial_u\}.$$

Proof. In (2) the variables u and x can have a scaling symmetry of the form $x\partial_x - u\partial_u$ and, if one counts the number of 'downs' it is $n+2$. Consequently for a balancing with a $t\partial_t$ term the coefficient must be $n+1$ as the t derivative must also be $n+2$ 'downs'.[2] □

3. Travelling-wave solutions

Reduction of (2) by either of Γ_1 or Γ_2 singly is not to be advised because that would remove one of the independent variables from the solution. However, the combination $\Gamma_1 + c\Gamma_2$, which gives a travelling-wave solution, makes sense. In terms of the new independent variable, $r = x - ct$, (2) becomes

$$D[(D+w)^n - c]w = 0, \tag{5}$$

where the new dependent variable is $w(r) = u(t, x)$ with $r = x - ct$ and D denotes differentiation with respect to r. When one expands (5), care must be taken with the normal ordering of the operators. For example in the cases $n = 1$ and $n = 2$ the reduced equations are

$$w'' + 2ww' - cw' = 0 \quad \text{and} \tag{6}$$

$$w''' + 3ww'' + 3w'^2 + 3w^2w' - cw' = 0, \tag{7}$$

respectively. The explicit absence of r from (5) indicates that the sole symmetry is ∂_r. The direct solution of (6) is fairly straight forward. That of (7) is not so simple but possible. However, a different approach enables a simple route to the solution for general n.

Theorem 3.1. *The general solution of (5) is*

$$w(r) = \frac{v'(r)}{v(r)},$$

[2] A self-similar symmetry in two variables is assumed to have the structure $\alpha t\partial_t + x\partial_x$. There are $n+2$ negative values following from the u_{n+1} term. As the time derivative is just u_t, the value of α must be $n+1$ because the u, although an 'up', has a negative coefficient.