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On the exact solution of the Riemann problem for blood flow in human veins, including collapse

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ABSTRACT

We solve exactly the Riemann problem for the non-linear hyperbolic system governing blood flow in human veins and note that, as modeled here, veins do not admit complete collapse, that is zero cross-sectional area A. This means that the Cauchy problem will not admit zero cross-sectional areas as initial condition. In particular, rarefactions and shock waves (elastic jumps), classical waves in the conventional Riemann problem, cannot be connected to the *zero state* with $A = 0$. Moreover, we show that the area A_* between two rarefaction waves in the solution of the Riemann problem can never attain the value zero, unless the data velocity difference $u_R - u_L$ tends to infinity. This is in sharp contrast to analogous systems such as blood flow in arteries, gas dynamics and shallow water flows, all of which admitting a *vacuum* state. We discuss the implications of these findings in the modelling of the human circulation system that includes the venous system.

1. Introduction

We are motivated by the practical aim of constructing global mathematical models for studying the biophysics of the human circulation system involving both arteries and veins, as well as the microvasculature. Governing equations for blood flow in both arteries and veins have been proposed in the literature. These involve averaged equations for conservation of mass and balance of momentum. To close the system, elastic tube laws have been proposed, which distinguish between arteries and veins. Most experience in the field relates to arteries [1–14]. For comprehensive reviews see [15,16] and the many references therein. Veins, as compared to arteries, have received much less attention; they are known to be highly non-linear, to exhibit large deformations, including collapse [17,18]. Simulating the venous system represents a challenge to the modeller and the designer of numerical algorithms for hyperbolic equations [19–22]. Important steps forward in this direction are reported in the recent works [23–28].

From the mathematical as well as numerical points of view, a basic problem to be solved is the special Cauchy problem called the Riemann problem [19]. For arteries, the exact solution was obtained in [29,30]; see also [27,31,32]. Interestingly enough, for the arterial case all classical cases are admitted, including collapsed states with zero cross-sectional area, or the *zero state*. Intriguingly, as shown in this paper, collapsed states are not admitted for veins. This is surprising and somehow contradictory, as under common physiological conditions, it is veins the ones that collapse. In fact, in two thirds of the

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normal population internal jugular veins are collapsed in the sitting position [33,34]. Arteries, on the other hand, under normal physiological conditions, are far from being collapsed, the internal pressure is high and the cross-sectional area is normally circular in shape. It is known that the cross-sectional area of veins is also circular in the high range of transmural pressures, becomes elliptical as the transmural pressure decreases and *virtually collapses* for large negative transmural pressures [17,18] but still preserving a non-zero (even if small) cross-sectional area. We take this limiting case as the *collapsed state for veins*, when the cross-sectional area assumes a buckled, dumbell shape configuration, in which opposite sides of the interior wall touch each other, still leaving some fluid in the two extremes. In other words, the model does not admit a zero cross-sectional area state. These observations are based on experiments [17,18,35]. In the present theoretical study some of these observations are replicated.

In this paper we succinctly report the exact solution of the Riemann problem for veins. We note that for the assumed equations and tube law, veins do not admit complete collapse, that is zero cross-sectional area A is never permitted, which means that the Cauchy problem will not admit zero cross-sectional areas as initial condition. In particular, neither rarefactions nor shock waves (elastic jumps) can connect a non-trivial state with $A > 0$ to a zero state with $A = 0$. Moreover, we prove that the area A_* between two rarefaction waves in the solution of the Riemann problem can never attain the value zero, unless the data velocity difference $u_R - u_L$ tends to infinity. This is in sharp contrast to analogous systems such as blood flow in arteries, gas dynamics and shallow water flows, all of which admitting a *vacuum* state.

The rest of the paper is structured as follows. In Section 2 we describe the one-dimensional blood flow equations with a general tube law, analyze the mathematical structure, state the solution of the Riemann problem and describe some properties of the solution of the Riemann problem. In Section 3 we present some selected examples of Riemann problems and discuss the results. In Section 4 we give a summary of the main results of this paper and discuss their implications.

2. Equations and the Riemann problem

2.1. Partial differential equations and tube laws

The governing one-dimensional, averaged equations for blood flow in compliant vessels are

$$\begin{cases} \partial_t A + \partial_x q = 0 \, , \\ \partial_t q + \partial_x \left(\alpha \dfrac{q^2}{A} \right) + \dfrac{A}{\rho} \partial_x p = s \, , \end{cases} \tag{1}$$

where $A(x, t)$ is the cross-sectional area of the vessel, $q(x, t)$ is the flow rate, $p(x, t)$ is the average internal pressure over a cross section, $s(x, t)$ is a source term, ρ is the blood density, assumed constant, and α is the Coriolis coefficient assumed $\alpha = 1$ here. The resulting system is hyperbolic and, despite the simplifications adopted in its derivation, preserves the fundamental physical aspects of wave propagation in compliant vessels. Full details on the derivation of the equations are found in [15] and [27], for example.

System (1) has more unknowns than equations, and hence an extra relation is required for closure. This is achieved by introducing an algebraic relation, called *tube law*, which relates pressure $p(x, t)$ to cross-sectional area $A(x, t)$. A purely elastic tube law reads

$$p(x,t) = p_e(x,t) + K(x)\psi(A(x,t); A_0(x)) \, , \tag{2}$$

with

$$\psi(A(x,t); A_0(x)) = \left[\left(\frac{A(x,t)}{A_0(x)} \right)^m - \left(\frac{A(x,t)}{A_0(x)} \right)^n \right] , \tag{3}$$

where $p_e(x, t)$ is the external pressure and $p_{trans} = p - p_e$ is the transmural pressure. $A_0(x)$ is vessel cross-sectional area at equilibrium (when $p_{tm} = 0$), $K(x)$ is the stiffness coefficient, $m \geq 0$ and $n \leq 0$ are real numbers to be specified. For hyperbolicity m and n must satisfy additional constraints, see [30]. For more information about the mathematical structure of the equations, see [15,27,30].

At this stage, one distinguishes between arteries and veins through the tube law, in particular through the coefficient of elasticity $K(x)$ and the exponents m and n, namely

$$K(x) = \begin{cases} K_a(x) = \dfrac{E(x)}{1-\nu^2} \dfrac{h_0(x)}{r_0(x)} \, , & m = \frac{1}{2} \, , \; n = 0 \quad \text{arteries,} \\ K_v(x) = \dfrac{E(x)}{12(1-\nu^2)} \left(\dfrac{h_0(x)}{r_0(x)} \right)^3 \, , & m \approx 10 \, , n = -\frac{3}{2} \quad \text{veins.} \end{cases} \tag{4}$$

Here ν, $h_0(x)$, $r_0(x)$, $E(x)$ are respectively the Poisson ratio (set to $\nu = 0.5$), wall-thickness at equilibrium, cross-sectional radius at equilibrium and Young's modulus of elasticity.

Collapsed state for veins. Note that for $n < 0$, function (3) is not defined at $A = 0$. This means that for veins, the *zero state* ($A = 0$) for the cross-sectional area is not permitted. We thus define a *collapsed state* as A_ϵ, with $0 < A_\epsilon < A_0$ a small, positive arbitrary real number, which in practical computations is to be chosen on machine accuracy considerations.