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# Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore–Penrose inverse

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ABSTRACT

Weighted singular value decomposition (WSVD) and a representation of the weighted Moore–Penrose inverse of a quaternion matrix by WSVD have been derived. Using this representation, limit and determinantal representations of the weighted Moore–Penrose inverse of a quaternion matrix have been obtained within the framework of the theory of noncommutative column-row determinants. By using the obtained analogs of the adjoint matrix, we get the Cramer rules for the weighted Moore–Penrose solutions of left and right systems of quaternion linear equations.



## 1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ be the real and complex number fields, respectively. Throughout the paper, we denote the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \,|\, i^2 = j^2 = k^2 = -1,\ a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m\times n}$, and by $\mathbb{H}_r^{m\times n}$ the set of all $m \times n$ matrices over $\mathbb{H}$ with a rank $r$. Let $\mathrm{M}(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices and $\mathbf{I}$ be the identity matrix with the appropriate size. For $\mathbf{A} \in \mathbb{H}^{n\times m}$, we denote by $\mathbf{A}^*$, $\operatorname{rank}\mathbf{A}$ the conjugate transpose (Hermitian adjoint) matrix and the rank of $\mathbf{A}$. The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The definitions of the generalized inverse matrices can be extended to quaternion matrices.

The Moore–Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$, denoted by $\mathbf{A}^\dagger$, is the unique matrix $\mathbf{X} \in \mathbb{H}^{n\times m}$ satisfying the following equations [1],

$$\mathbf{AXA} = \mathbf{A}; \tag{1}$$

$$\mathbf{XAX} = \mathbf{X}; \tag{2}$$

$$(\mathbf{AX})^* = \mathbf{AX}; \tag{3}$$

$$(\mathbf{XA})^* = \mathbf{XA}. \tag{4}$$

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Let Hermitian positive definite matrices $\mathbf{M}$ and $\mathbf{N}$ of order $m$ and $n$, respectively, be given. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, *the weighted Moore–Penrose inverse* of $\mathbf{A}$ is the unique solution $\mathbf{X} = \mathbf{A}^{+}_{M,N}$ of the matrix Eqs. (1) and (2) and the following equations in $\mathbf{X}$ [2]:

$$(3M)\ (\mathbf{MAX})^* = \mathbf{MAX};\ \ (4N)\ (\mathbf{NXA})^* = \mathbf{NXA}.$$

In particular, when $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{N} = \mathbf{I}_n$, the matrix $\mathbf{X}$ satisfying the Eqs. (1), (2), $(3M)$, and $(4N)$ is the Moore–Penrose inverse $\mathbf{A}^{\dagger}$.

Generalized inverses and their using to solutions of matrix equations over the quaternion skew field (or arbitrary non-division ring) are subjects of current research (see, e.g., [3–12]).

It is known various representations of the weighted Moore–Penrose. In particular, limit representations have been considered in [13,14]. Determinantal representations of the complex (real) weighted Moore–Penrose have been derived by full-rank factorization in [15], by limit representation in [16] using the method first introduced in [17], and by minors in [18]. A basic method for finding the Moore–Penrose inverse is based on the singular value decomposition (SVD). It is available for quaternion matrices, (see, e.g. [24,25]). In [25,27], using SVD of quaternion matrices, the limit and determinantal representations of the Moore–Penrose inverse over the quaternion skew field have been obtained within the framework of the theory of noncommutative column-row determinants that have been introduced in [28].

In [29], the weighted Moore–Penrose inverse $\mathbf{A}^{\dagger}_{M,N} \in \mathbb{C}^{m\times n}$ can be explicitly expressed by the weighted singular value decomposition (WSVD) which at first has been obtained by Cholesky factorization. In [30], WSVD of real matrices with singular weights has been derived using weighted orthogonal matrices and weighted pseudoorthogonal matrices.

Song at al. [31] and Song and Wang [32] have studied the weighted Moore–Penrose inverse over the quaternion skew field and obtained its determinantal representation within the framework of the theory of column-row determinants. But WSVD of quaternion matrices has not been considered and for obtaining a determinantal representation there was used auxiliary matrices which different from $\mathbf{A}$, and weights $\mathbf{M}$ and $\mathbf{N}$.

The main goals of the paper are introducing WSVD of quaternion matrices and representation of the weighted Moore–Penrose inverse over the quaternion skew field by WSVD, and then by using this representation, obtaining its limit and determinantal representations. Using the obtained analogs of the adjoint matrix, we plan to derive the Cramer rules for the weighted Moore–Penrose solutions of left and right systems of quaternion linear equations.

In this paper we shall adopt the following notation.

Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \le k \le \min\{m, n\}$. By $\mathbf{A}^{\alpha}_{\beta}$ denote the submatrix of $\mathbf{A}$ determined by the rows indexed by $\alpha$, and the columns indexed by $\beta$. Then, $\mathbf{A}^{\alpha}_{\alpha}$ denotes the principal submatrix determined by the rows and columns indexed by $\alpha$. If $\mathbf{A} \in \mathrm{M}(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}^{\alpha}_{\alpha}|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \le k \le n$, denote by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k),\ 1 \le \alpha_1 \le \dots \le \alpha_k \le n\}$ the collection of strictly increasing sequences of $k$ integers chosen from $\{1, \dots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let

$$I_{r,\,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}, \quad J_{r,\,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}.$$

The paper is organized as follows. We start with some basic concepts and results from the theory of the row-column determinants and of Hermitian quaternion matrices in Section 2. Weighted singular value decomposition and a representation of the weighted Moore–Penrose inverse of quaternion matrices by WSVD have been considered in Section 3.1, and its limit representations in Section 3.2. In Section 4, we give the determinantal representations of the weighted Moore–Penrose inverse when the matrices $\mathbf{N}^{-1}\mathbf{A}^*\mathbf{MA}$ and $\mathbf{AN}^{-1}\mathbf{A}^*\mathbf{M}$ are Hermitian in Section 4.1, and when they are non-Hermitian in Section 4.2. In Section 5, we obtain explicit representation formulas of the weighted Moore–Penrose solutions (analogs of Cramer's rule) of the left and right systems of linear equations over the quaternion skew field. In Section 6, we give numerical examples to illustrate the main result. Finally, a brief conclusion is given in Section 7.

## 2. Preliminaries

For the first time, the theory of column-row noncommutative determinants (i.e. determinants with noncommutative entries) has been introduced in [26] (in Russian) and as amended in [27].

For a quadratic matrix $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ can be define $n$ row determinants and $n$ column determinants as follows.

Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \dots, n\}$.

**Definition 2.1** [27]**.** The $i$th row determinant of $\mathbf{A} = (a_{ij}) \in \mathrm{M}(n, \mathbb{H})$ is defined for all $i = \overline{1, n}$ by putting

$$\mathrm{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i\, i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i}) \dots (a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}}),$$

$$\sigma = \left(i\, i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}\right) \dots \left(i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}\right),$$

with conditions $i_{k_2} < i_{k_3} < \dots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.