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High-order implicit finite difference schemes for the two-dimensional Poisson equation

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ABSTRACT

In this paper, a new family of high-order finite difference schemes is proposed to solve the two-dimensional Poisson equation by implicit finite difference formulas of $(2M+1)$ operator points. The implicit formulation is obtained from Taylor series expansion and wave plane theory analysis, and it is constructed from a few modifications to the standard finite difference schemes. The approximations achieve $(2M+4)$-order accuracy for the inner grid points and up to eighth-order accuracy for the boundary grid points. Using a Successive Over-Relaxation method, the high-order implicit schemes have faster convergence as M is increased, compensating the additional computation of more operator points. Thus, the proposed solver results in an attractive method, easy to implement, with higher order accuracy but nearly the same computation cost as those of explicit or compact formulation. In addition, particular case $M=1$ yields a new compact finite difference schemes of sixth-order of accuracy. Numerical experiments are presented to verify the feasibility of the proposed method and the high accuracy of these difference schemes.

1. Introduction

In this paper, we seek high-order accuracy numerical solutions of the Poisson equation

$$\Delta p(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{1}$$

where Ω is a two-dimensional rectangular domain with Dirichlet boundary conditions defined on $\partial\Omega$, and Δ is the Laplacian operator. The solution p and the forcing function f are assumed to be sufficiently smooth and to have the required continuous partial derivatives. When $f(x,y)=0$, Eq. (1) becomes the Laplace equation.

The Poisson and Laplace equations are partial differential equations with broad applications in different fields, such as computational fluid dynamics, Structural Mechanics, theoretical physics, etc. Eq. (1) is often used to describe equilibrium phenomena for many variables such as pressure, water surface elevation, temperature and concentration [1] and its numerical solution is fundamental for the computational simulation of the corresponding applied problems. A large amount of work has been done in the past decades to develop numerical methods for the solution of the Poisson equation producing accurate results. It is known that the standard second-order finite difference method has to take small mesh sizes for obtaining desirable accuracy. Therefore, in order to obtain satisfactory numerical results with reasonable computational

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cost, a reasonable approach is to develop a higher-order finite difference method. As one of the most effective numerical implementations, the fourth-order compact nine-point-finite difference schemes have received much attention due to their advantages in solution accuracy and the time efficiency. We refer to [2–7], among others.

On the contrary, few sixth-order schemes have been developed to compute the solution. Previously, Chu and Fan [8] proposed a three point combined compact difference scheme with a Hermitian polynomial approximation for solving a special two dimensional convection diffusion equation. Sun and Zhang [9] proposed a sixth-order explicit finite difference discretization strategy for solving the 2D convection diffusion equation based on Alternating Direction Implicit Method and the Richardson Extrapolation Technique. Nabavi et al. [10] proposed a new sixth-order accurate compact finite difference method for solving the Helmholtz equation. The work of Wang and Zhang [11] derived an implicit sixth-order compact finite difference scheme for 2D Poisson equation using Taylor series expansions. One of the most recent sixth-order discretization method was proposed by Zhai et al. [12]. They choose a special dual partition and employ Lagrange interpolation and Simpson integral formula to derive difference schemes.

In this work, we developed high-order finite difference schemes (fourth-, sixth-, eighth-order, etc.) by using the idea of implicit schemes and any number of stencil operator points for resolving the Poisson Eq. (1). The scheme is referred to as an implicit high order compact scheme because the first derivative and the second derivative of the dependent variables are computed at the same time.

The algorithms that we propose have several advantages. First, the implicit finite difference scheme is accurate up to the eight-order and thus provides precise numerical results. Moreover, the scheme is more accurate if the solution of the problem is not significantly affected by the boundary conditions. The general algorithm is not a compact scheme, however the proposed high-order implicit schemes have a better rate of convergence as more operator points are considered. As a consequence, the smaller number of iteration steps compensates for the required computational time generated by the arithmetic operations associated with the additional nodes. Thereby making our algorithm competitive in comparison with the compact schemes for solving the Poisson equation. Furthermore, compact schemes of fourth- and sixth-order can be derived as special cases. Lastly, the proposal scheme is written such that an exact knowledge of the forcing function and its corresponding partial derivatives is not required, as other authors' formulas [12]. Instead, the scheme requires the evaluation of the forcing function at specific mesh points, as it happens in most practical cases.

This paper is organized as follows. The implicit finite difference formulas will be derived first. Subsequently, the accuracy of the scheme will be analyzed. The high-order implicit finite difference scheme for the Poisson equation will be described in Section 3. In Section 4, two numerical examples will be presented to examine the accuracy and effectiveness of the proposed scheme. The results will be compared with those from the explicit standard difference and compact nine-point-difference schemes. Finally, conclusions will be provided in Section 5.

2. Implicit finite difference formulas

This section shows the approximations of the second-order derivative of univariate functions p by implicit finite differences of high orders based on classical plane wave theory. These approximations will be used to obtain finite difference schemes of high-order for the one- and two-dimensional Poisson equation.

2.1. Explicit finite difference formula

Let us consider the classic one-dimensional finite difference approximation for a second derivative of a real-valued function p. Given a small value $h > 0$, the second-order derivative at x can be approximated by the following centered finite difference operator

$$\delta_x^2 p = \frac{1}{h^2}[p(x+h) - 2p(x) + p(x-h)], \tag{2}$$

where the approximation error is to the second-order of accuracy. In order to improve the accuracy of (2), we define an approximation to the second derivative by taking more points with appropriate weighting. Thus for a fixed positive integer M, the second-order derivative of a function p can be approximated by a formula of $2M+1$ nodes as

$$\frac{\partial^2 p}{\partial x^2}(x) = \frac{\delta^2 p}{\delta x^2}(x) + O(h^{2M}), \tag{3}$$

where the *explicit finite difference* (EFD) of $(2M)$th-order of accuracy is given by

$$\frac{\delta^2 p}{\delta x^2}(x) = \frac{1}{h^2}\sum_{m=-M}^{M} c_m p(x+mh), \tag{4}$$

where values c_m are finite difference coefficients which can be easily determined using wave plane theory [13]. For each number $m \in \{1, 2, \dots, M\}$, coefficients c_m and c_{-m} correspond to the same values, resulting in a symmetric finite difference formula (4).